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Surveying

BOOK II

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TRANSIT SURVEYING
OFFICE WORK IN ANGULAR SURVEYING
CIRCULAR AND PARABOLIC CURVES
Parts 1–2

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TRANSIT SURVEYING

THE TRANSIT

Serial 3067–3 Edition 1

PRELIMINARIES

DESCRIPTION OF INSTRUMENT

1. Introduction.—The engineer's, or surveyor's, transit is used almost exclusively to measure horizontal and vertical angles in surveying because it combines the features of convenience and a high degree of accuracy. By means of a telescope, the line of sight is well defined and long sights may be taken. Moreover, by the aid of verniers, readings on the graduated scales can be made very accurately. Although the transit is primarily intended for measuring angles without reference to the magnetic needle, most transits have a magnetic needle and a graduated needle circle, and may, therefore, be used as a compass.

While there are many kinds of engineer's transits, all are constructed on the same principles, differing only in minor details. Two forms of transits are shown in Figs. 1 and 2.

2. Telescope.—The telescope *a*, Fig. 1, is similar to that on an engineer's level but is shorter in length; its parts are the objective *b*, the focusing wheel *c*, the cross-wires at *d*, the eyepiece *e*, and the sunshade *f*. The telescope is fixed to the axis *g*, called the *transverse axis*, which rests on the standards *h* and revolves in bearings at the top of the standards. An important feature of the transit shown in Fig. 2 is the rigid construction

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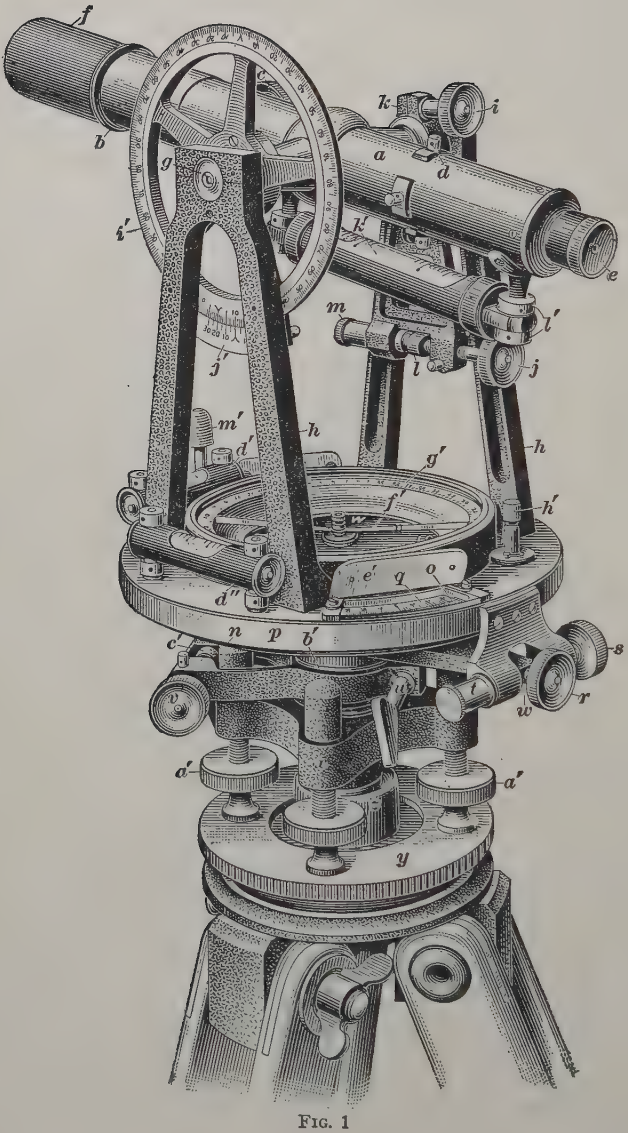


FIG. 1

of the standards, which are cast in one piece. The telescope is held at any desired inclination by means of the clamp screw *i*, Fig. 1, and can be rotated slowly in a vertical plane by means of a tangent screw *j* attached to one standard. The clamp *i* passes through a collar *k* in which the axis *g* revolves when the screw *i* is loose; and a projection *l*, attached to the collar through an arm, fits between the point of the screw *j* and the opposing spring *m*. When the clamp *i* is tightened, the axis *g* is held firmly in the collar *k* and is prevented from moving. If the screw *j* is then turned so that its point pushes against the projection *l*, the axis *g* is caused to rotate in one direction, the

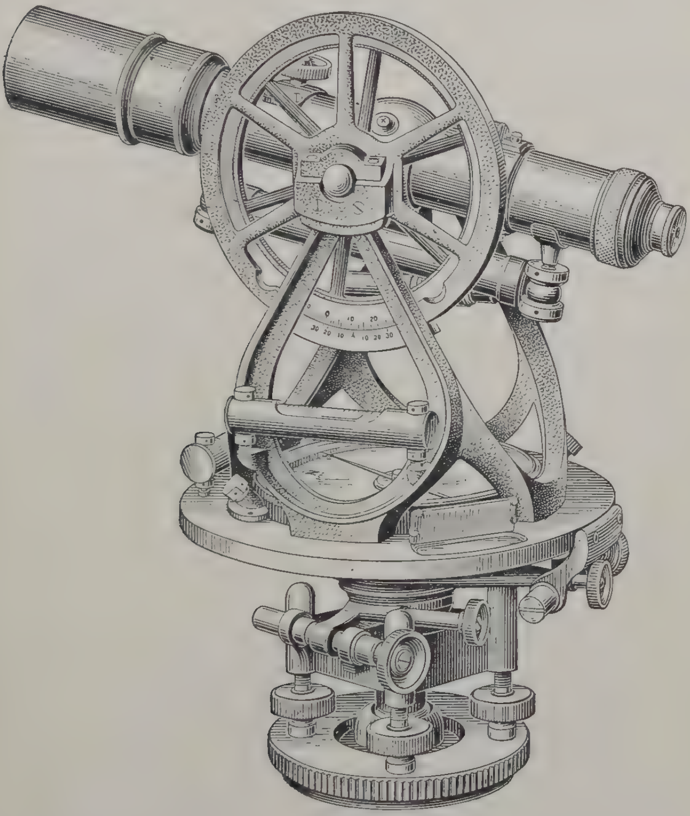


Fig. 2

spring *m* being compressed. When the screw *j* is turned back, the spring forces the projection *l* against the point of the screw and produces rotation of the axis *g* in the opposite direction.

3. Plates and Centers.—In order that horizontal angles may be measured, all transits have two concentric plates, which rotate independently on the same axis, called the *axis of the instrument.* On the lower of these plates *n*, Fig. 1, is a graduated scale, called the *horizontal limb* or the *horizontal circle*, a small part of which is shown at *o*; and on the upper plate *p* are two verniers, one of which is shown at *q*. The upper plate also carries the standards, which are fixed to it.

The upper plate is held in position with respect to the lower plate by tightening the clamp screw *r*, called the *upper clamp;* this operation is known as *clamping the upper plate.* After the upper plate has been clamped, it can be revolved slowly through a small angle by means of the *upper tangent screw s*, which operates against the spring *t*. The lower plate may be secured or clamped against rotation by tightening the *lower clamp u;* this is called *clamping the lower plate.* When the lower plate is clamped, slow motion may be obtained by using the *lower tangent screw v*, which works against a spring. Before either tangent screw is used, the corresponding clamp must be tightened.

The construction of the lower part of a transit is shown in Fig. 3, which represents a cross-section through the axis of the instrument. The conical spindle *n'*, which is connected to the upper plate *p*, Figs. 1 and 3, revolves within a socket attached to the lower plate *n*; the spindle and the socket are held by the nut *o'*, Fig. 3, which screws on the bottom of the spindle. The upper clamp *r*, Figs. 1 and 3, passes through the projection *w* on the upper plate, which fits between the point of the screw *s* and the spring *t*. When the screw *r* is tightened, the collar *p'*, Fig. 3, is pressed against the lower plate and the upper plate is thus secured against rotation on the lower. Slow motion is then obtained by means of the tangent screw *s*.

The socket on the lower plate fits inside of another socket in the *leveling head x*, Figs. 1 and 3. This combination of the spindle and sockets is called the *centers*. The leveling head is connected to the *tripod plate* y by a flexible joint as shown at z, and the inclination of the leveling head and its socket is controlled by the leveling screws a', which bear on the plate y.

The lower clamp and tangent screw pass through the collar b', and the projection c' from the leveling head fits between the point of the screw v and the opposing spring. When the clamp u is tightened, the lower plate cannot turn. By means of the tangent screw v, however, slow motion can be produced.

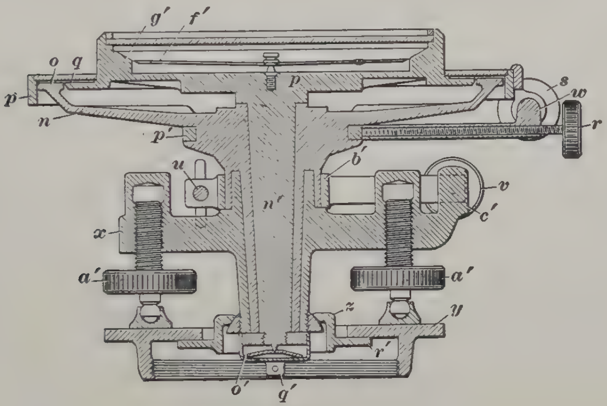


FIG. 3

The spirit levels d' and d'', Fig. 1, called *plate levels*, are placed at right angles to each other. When each bubble is in the center of its tube at the same time, the plates are horizontal. In Fig. 2, one of the levels is on the standard, but often both are attached to the upper plate, as in Fig. 1.

4. Scales.—Horizontal angles are measured by means of the horizontal limb o, Figs. 1 and 3, marked on the edge of the lower plate, and the verniers q, attached to the upper plate. The upper plate entirely conceals the lower plate except at two small glass-covered openings for the verniers, which are diametrically opposite. The surface of the *reflector* e', Fig. 1, is specially prepared to reflect light to the vernier and the circle. On the transit shown in Fig. 2, the reflector is hinged so that it can be dropped over the glass when the instrument is not in use, or can be set to any desired inclination.

On the upper plate there is also a magnetic needle f', Figs. 1 and 3, and a needle circle g' for use when magnetic bearings are taken. The circle is graduated in the same way as that on a compass, and the north and south points are directly under the line of sight. The screw h', Fig. 1, is for lifting the needle off its pivot.

5. The angle at which the telescope is inclined to the horizontal for any position is measured by means of the graduated scale i', Fig. 1, called the *vertical limb*, and the vernier j'. Some transits have an arc, called a *vertical arc*, instead of a full vertical circle as in Fig. 1, while others do not have any vertical limb and do not measure vertical angles. The vertical limb is attached to the transverse axis and revolves with it; the vernier j' is screwed to one of the standards. In Fig. 2, the graduated edge of the vertical limb is protected by a guard around it.

A spirit level k', Fig. 1, called a *telescope level*, is attached to the telescope longitudinally whenever there is an arrangement for measuring vertical angles; the telescope level permits the transit to be used also as a leveling instrument. The level tube, which has a graduated scale similar to that on an engineers' level, can be adjusted vertically with respect to the telescope by means of capstan-pattern nuts l' at each end; no lateral adjustment is necessary. The rubber stop m' prevents the telescope from striking the plate level.

A transit without a vertical limb or a telescope level is called a *plain transit*.

6. Graduations.—The horizontal limb is graduated in various ways, with respect both to the size of the smallest divisions and to the method of numbering. The degrees are marked on all instruments but the number of parts into which each degree is divided and the number of vernier divisions vary. The most common method is to graduate the limb in half degrees and to divide the vernier into 30 parts covering 29 of these half-degree divisions; then, readings can be taken accurately to the nearest minute. On some instruments, the limb is graduated in 20-minute spaces and the vernier is divided into 40 parts; in this case, angles can be measured to 30 seconds. For very precise work, there are instruments whose limbs and verniers are graduated to read to 20 seconds, 10 seconds, and even 5 seconds.

Each subdivision of a degree is marked by a line somewhat shorter than the regular degree-graduations, while each fifth degree is indicated by a longer division line and each tenth degree is marked by a still longer line and also is numbered. Three systems of numbering are in common use, each having its advantage for certain kinds of work. These systems may be described as follows:

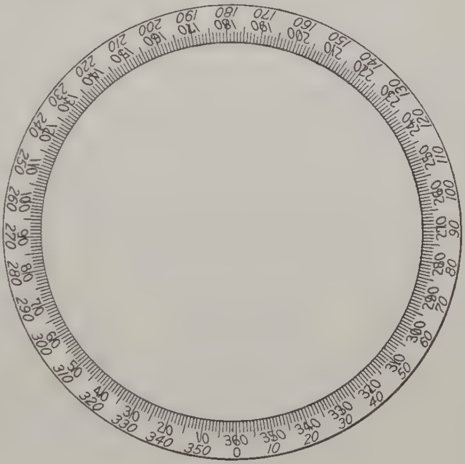


Fig. 4

1. The *azimuth system*, in which the graduation marks are numbered from 0 continuously around the entire circle to 360.

2. The *transit system*, in which the figures extend from 0 in opposite directions through the adjacent semicircles to 180 at the point diametrically opposite the zero point.

3. The *compass system*, in which the figures extend each way from two 0 points diametrically opposite each other through the adjacent quadrants to the 90° points.

Usually the horizontal limb of a transit has two sets of figures, either of which may be used independently of the other. Ordinarily, the numbers of one set increase from 0 to 360 in a clockwise direction and often the numbers of the other set run from 0 to 360 in the opposite direction, as shown in Fig. 4. However, the other systems are also used, as in Figs. 5 and 6.

On most transits, the direction in which the numbers increase is indicated by the inclination of the figures, as shown in Figs. 4, 5, and 6.

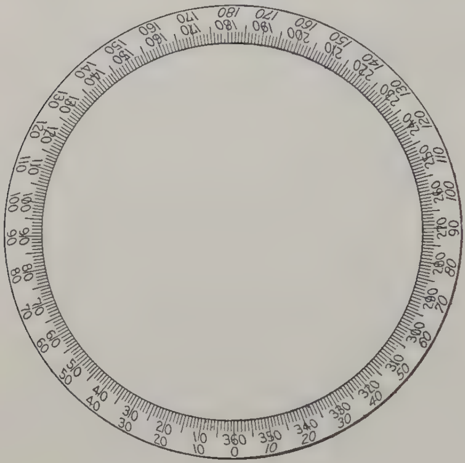


Fig. 5

The vertical limb is usually graduated to degrees and half degrees and, by means of the vernier, readings can be taken to the nearest minute. The vertical circle is always numbered in quadrants with the zero mark opposite the zero of the vernier when the telescope is horizontal. There is only a single row of numbers.

7. Shifting Head.—The position of the instrument over a point on the ground is indicated by a plumb-bob suspended from the lower end of the centers at the central point *q*′, Fig. 3. When the leveling screws *a*′ are tight, the plate *r*′ is held firmly against the plate *y*; but when the leveling screws are loose, the rest of the instrument drops with respect to the plate *y* and can be shifted on it. This arrangement, which is called a *shifting head*, is a great advantage in setting up the transit exactly over a point.

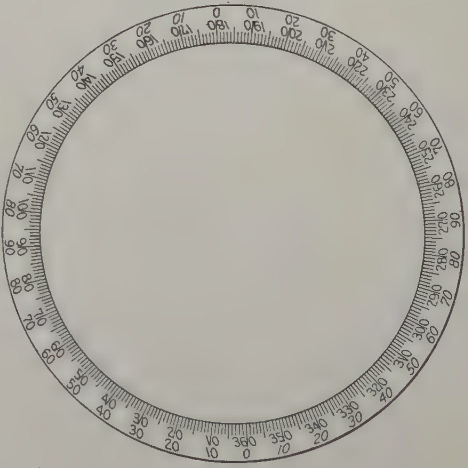
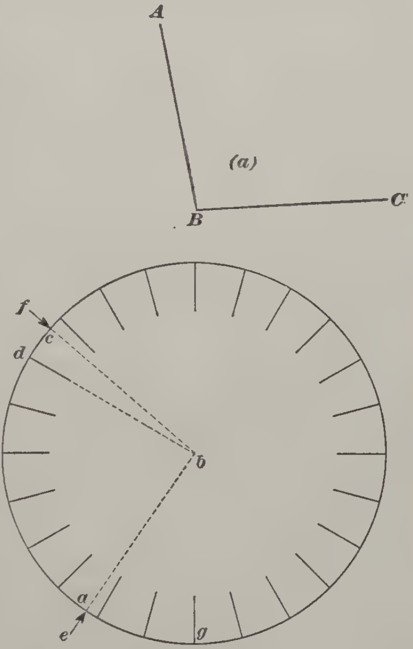


Fig. 6

READING OF VERNIER

8. Measurement of Angles.—In order to measure the angle ABC, Fig. 7 (a), the transit is set at B, the telescope is directed to A, the upper clamp is loosened, and finally the telescope is directed to C. While the line of sight rotates from the direction BA to the direction BC, the horizontal limb of the transit remains stationary and the verniers move along the limb. The amount of the movement in degrees of arc is, therefore, the size of the angle. In order to determine this amount, the location of the zero point of the vernier on the horizontal limb is observed for each position of the telescope.



(b)

FIG. 7

In Fig. 7 (b), let adc represent the horizontal limb; b, the center of the graduated circle; e, the zero point of the vernier when the telescope is directed along BA in (a); f, the zero point of the vernier when the line of sight is directed along BC in (a); and g, the zero point of the horizontal limb. Then, the arc ac in (b), which measures the angle ABC in (a), is found by subtracting the reading of the limb at a from the reading of the limb at c. For convenience the vernier is usually set to read zero when the line of sight is directed to A.

9. Use of Verniers.—One of the features which gives a transit its great accuracy is the fact that fractional parts of the smallest division of the limb can be measured by means of a vernier. Although the limb of a transit is graduated in angular measure and the scale is marked along a curved edge, the principles of the transit vernier are the same as those of the vernier described in connection with leveling rods. For instance, in Fig. 7 (*b*), the arc *g d c* is considered to be composed of the arc *g d*, measured on the limb, and the arc *d c*, measured by the vernier.

Suppose that, in Fig. 8, *A B* represents a circular scale and *CD* a vernier that slides along the scale. To measure the arc *EF*, the zero point of the scale *AB* is set at *E* and the vernier is placed so that its zero mark is at *F*. The arc *EF* is composed of two parts: the portion *EG* from the zero of the scale to the scale graduation at *G* just preceding the zero of the vernier, and the part *GF*. The zero of the vernier is generally known as the *index* of the vernier, and the reading of the scale opposite the index of the vernier is called the *reading of the vernier*. To determine the reading of a vernier, three steps are performed: (1) the value of the limb graduation preceding the index of the vernier is observed; (2) the value of the fractional part of a division from that graduation to the index of the vernier is found by means of the vernier; and (3) the two values thus obtained are added.

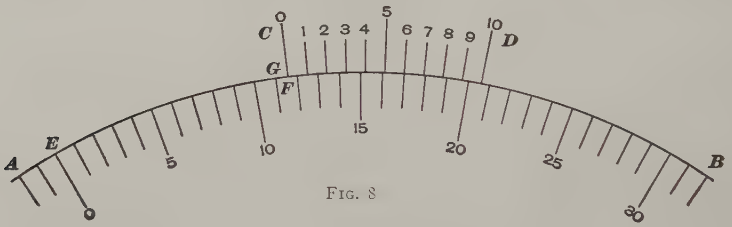


FIG. 8

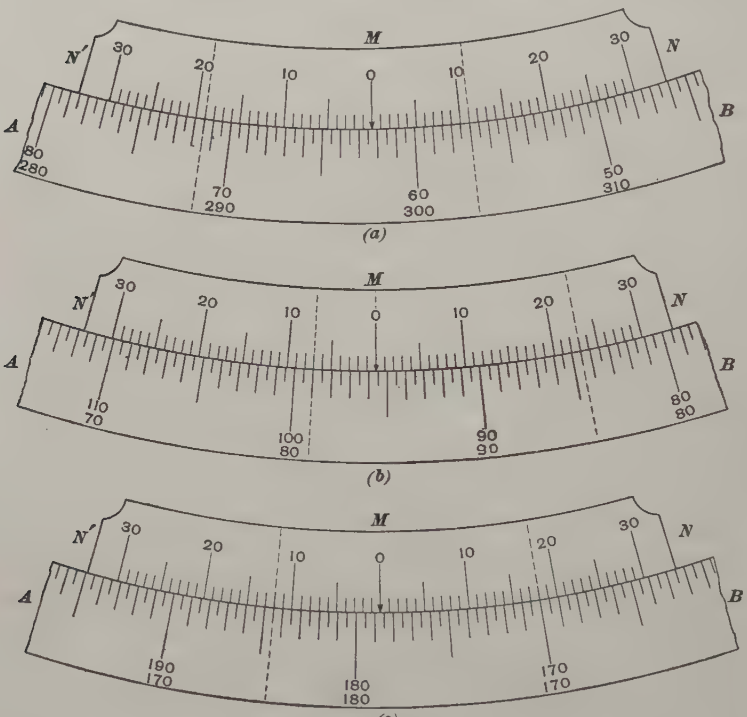
10. Suppose that the limb *AB*, Fig. 8, is divided in degrees and it is required to measure the arc to tenths of a degree. In this case, the vernier is made with a length equal to 9 limb divisions, and is divided into 9+1, or 10, equal parts. Then, the total length of the vernier is 9×1, or 9, degrees, and each vernier division is equal to $\frac{1}{10}\times 9=\frac{9}{10}$ degree. Hence, the difference between one limb division and one vernier division is $1-\frac{9}{10}=\frac{1}{10}$ degree. This difference, which is called the *least reading of the vernier*, is always found by dividing the value of a limb division by the number of parts into which the vernier is divided. Thus, in Fig. 8, the least reading of the vernier is equal to $\frac{1}{10}$ degree because each limb division is 1 degree and there are 10 parts in the vernier.

Since the graduation at G is one beyond the mark numbered 10, the arc EG represents 11 degrees. The value of the arc GF is equal to the product of the least reading of the vernier and the number of the vernier graduation that coincides with a graduation on the limb. Since, in this case, the graduation numbered 6 is opposite a line on the limb, the arc GF is equal to $\frac{6}{10}$ degree. This may be proved as follows: The distance from graduation 5 on the vernier to graduation 16 on the limb is equal to the difference between one division on the limb and one on the vernier, or $\frac{1}{10}$ degree; the distance between graduation 4 on the vernier and graduation 15 on the limb is equal to the difference between two limb divisions and two vernier divisions, and so on; finally, the distance from the index of the vernier, which is opposite point F, to the limb graduation preceding it, is equal to the difference between six divisions of the limb and six of the vernier, or $\frac{6}{10}$ degree. Hence, the arc EF is $11+\frac{6}{10}$, or $11\frac{6}{10}$, degrees.

11. As in the case of a vernier on a leveling rod, the number of parts into which a transit vernier is divided is one more than the number of limb divisions covered by the vernier. Moreover, the numbers on the vernier increase in the same direction as do those on the limb. Then the value of the part of a division from a limb graduation to the index is found by multiplying the least reading of the vernier by the number of the vernier graduation which coincides with a limb graduation. The number of the limb graduation with which the vernier graduation coincides is not observed.

The numbers of the graduations on the limb of a transit do not increase in the same direction on all parts of the limb, and, therefore, as shown in Fig. 9, the vernier on a transit consists of two similar parts, the numbers of the graduations increasing in both directions from the index; one half of the vernier is used at a time, as will be explained in the following articles.

12. The limb *AB* in Fig. 9 (*a*), (*b*), and (*c*) is graduated to half degrees; and each part of the vernier *NN'* has 30 divisions, which cover 29 divisions on the limb. Hence, the least reading of the vernier, which is equal to the value of a limb division divided by the number of parts in the vernier, is $\frac{30'}{30} = 1$ minute. In (*a*) is shown part of a limb on which the graduations are numbered from 0 to 360 in both directions. The figures on the limb and on the vernier of a transit are usually inclined, but they are shown upright here. If the



(*a*)

(*b*)

(*c*)

FIG. 9

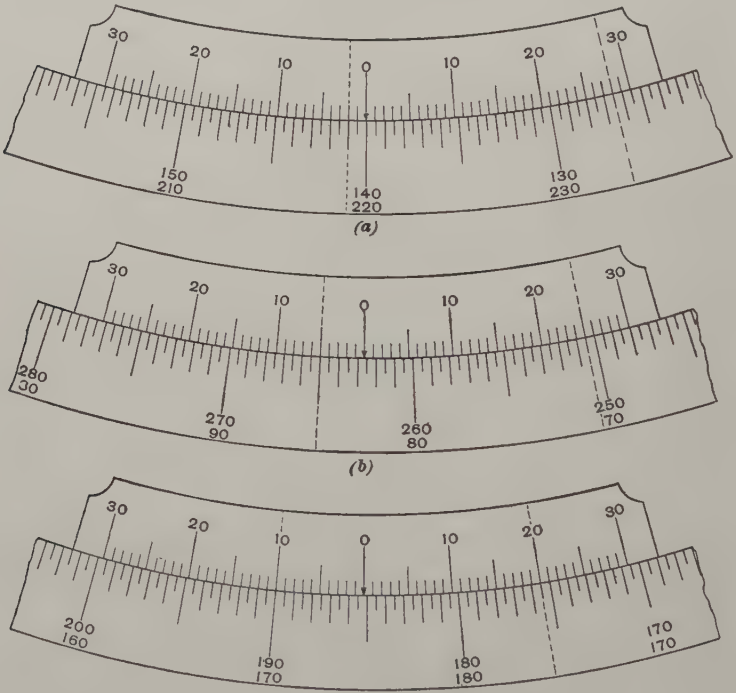
inner scale is read, the numbers increase in a clockwise direction and, therefore, the graduation preceding the index is 62°. As the numbers on the vernier must increase in the same direction as the numbers on the main scale, the vernier from M to N' is used for readings on the inner scale. The nineteenth vernier division coincides with a graduation on the limb, and the least reading of the vernier is 1 minute; hence, the reading of the vernier, which is equal to the product of the least reading and the number of the coinciding vernier graduation, is 1×19, or 19, minutes. The reading of the limb on the inner scale in Fig. 9 (*a*) is, therefore, 62° 19′.

If the outer scale is used, the numbers increase in a counter-clockwise direction and the vernier from M to N is employed. In Fig. 9 (*a*) the graduation preceding the index is 297° 30′ and the vernier reads 11 minutes. Hence, the reading of the limb is 297° 30′+11′, or 297° 41′. It is a common mistake to call the reading 297° 11′; care must, therefore, be taken to include the 30 minutes.

A vernier, such as NN', in which the divisions are numbered in both directions from the center, is known as a *double vernier*. On each transit there are two double verniers, which should be exactly 180° apart on the limb.

13. In Fig. 9 (*b*) is shown part of a limb numbered clockwise from 0 to 360 on the inside, and according to the quadrant system on the outside. The reading for the inner numbers is 95° 30′+7′, or 95° 37′, because the graduation preceding the index is 95° 30′ and the seventh division of the vernier MN' coincides with a graduation on the limb. For the outer set, the numbers between which the index lies increase in a counter-clockwise direction; the reading is 84° 23′, the vernier MN being used. Care must be taken to notice the numbers on each side of the index so that the degrees are not counted in the wrong direction and from the wrong graduation. For instance, the reading for the outer set might be called 95° 37′ by counting clockwise from 90 instead of counter-clockwise from 80.

In Fig. 9 (*c*), AB is part of a limb on which the numbers of the inner set run clockwise from 0 to 360 and those of the outer set run from 0 to 180 in both directions. Since the index lies between 170° and 180° for both sets of numbers, the reading in this case is the same whether the inner or the outer set is used. The value is $178° 30' + 12' = 178° 42'$. Here, there is a chance of reading in the wrong direction from 180; that is, the reading may be called 181° 18′ by counting counter-clockwise from 180 instead of clockwise from 170.



(a)

(b)

(c)

FIG. 10

EXAMPLE FOR PRACTICE

The verniers shown in Fig. 10 read to minutes; verify the following readings:

	INNER	OUTER
(*a*)	140° 2′	219° 58′
(*b*)	262° 35′	82° 35′
(*c*)	185° 10′	174° 50′

AZIMUTHS

14. Forward and Back Azimuths.—Just as a line has a forward and a back bearing, it also has a forward and a back azimuth; that is, the azimuth of a line in one direction is its *forward azimuth* and the azimuth of the same line in the opposite direction is its *back azimuth*. But when the term azimuth is used, the forward azimuth is meant. In Fig. 11, let *AB* be a line whose azimuth is 115°; *NS*, the meridian at *A*; and *N′S′*, parallel to *NS*, the meridian at *B*. Then the angle *NAB* is 115° and the angle *N′BB′*, between the meridian and the prolongation of *AB*, is also 115°. The azimuth of *BA*, which is equal to the angle *N′BA* measured clockwise, is 115°+180°, or 295°. This is also the back azimuth of *AB*. If the azimuth of *BA* is given as 295°, then the angle *N′BA* is 295°, as shown; evidently, the angle *NAA′* measured clockwise is also equal to 295°. Hence, the azimuth of *AB*, which is equal to the angle *NAB*, is 295°−180°, or 115°. This is likewise the back azimuth of *BA*.

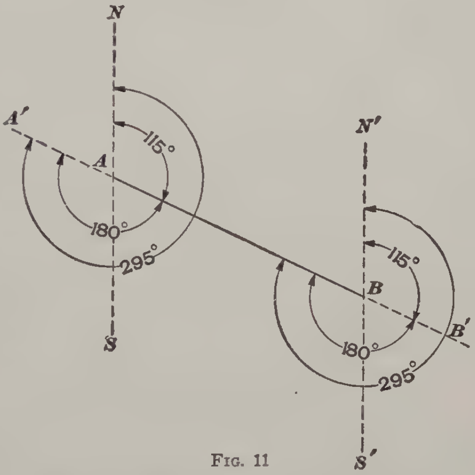


FIG. 11

15. From the preceding explanation, it is seen that the back azimuth of a line can be found from its forward azimuth by one of the following rules:

Rule I.—*If the azimuth of a line is less than 180°, add 180° to find the back azimuth.*

Rule II.—*If the azimuth of a line is greater than 180°, subtract 180° to obtain the back azimuth.*

For example, if the azimuth of a line is 75°, its back azimuth is equal to $75°+180°=255°$. If the azimuth of a line is 246°, the back azimuth is $246°-180°=66°$.

16. Angles Between Lines.—A line from which an angle is measured in surveying is called a *backsight;* the other side of the angle is a *foresight.* For example, when the angle BAC, Fig. 12 (a), is measured from AB to AC, the backsight is AB and the foresight is AC; likewise, if the angle DAE in (b) is measured from AD, the backsight is AD and the foresight is AE. Sometimes the points to which sights are taken in measuring an angle are also referred to as the backsight and the foresight.

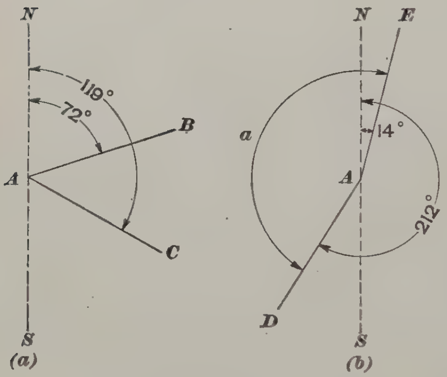


Fig. 12

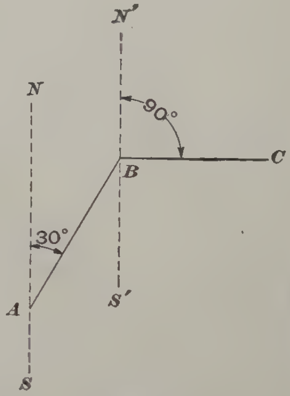


Fig. 13

Suppose that the azimuths of the lines AB and AC in (a) are 72° and 119°, respectively; then, if NS is the meridian through A, the angle NAB is 72° and the angle NAC is 119°. The angle BAC is obviously equal to the difference between NAC and NAB, which is $119°-72°=47°$; in other words, the angle between AB and AC is equal to the difference between their azimuths. Now, suppose it is required to find the angle a between the lines AD and AE in (b), the azimuths being, respectively, 212° and 14°. The angle DAN is equal to $360°-212°$ and the angle NAE is 14°. Hence, the angle DAE, which is equal to the sum of the angles DAN and

NAE, is $360° - 212° + 14° = 360° + 14° - 212° = 162°$. Since, in this case, the azimuth of the foresight *AE* is less than that of the backsight *AD*, it is necessary to add 360° to the azimuth of *AE* before the azimuth of *AD* is subtracted from it.

From the preceding explanations, the following rules can be given for finding the angle between two lines when their azimuths are known.

Rule I.—*The angle between two lines, measured clockwise, is equal to the azimuth of the foresight minus the azimuth of the backsight; but in case the azimuth of the foresight is less than that of the backsight, 360° is added to the smaller value before the larger is subtracted from it.*

Rule II.—*The angle between two lines, measured counter-clockwise, is equal to the azimuth of the backsight minus the azimuth of the foresight, 360° being added to the former if necessary before the subtraction is performed.*

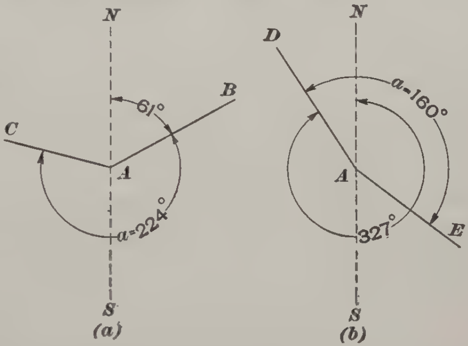


FIG. 14

In computing the angle between two lines from their azimuths, both the backsight and the foresight must be taken as starting at the vertex of the angle. For example, let it be required to determine the angle *ABC*, Fig. 13, the azimuths of *AB* and *BC* being 30° and 90°, respectively. The angle *ABC*, counter-clockwise, is equal to the azimuth of *BA* (not *AB*) minus the azimuth of *BC*. The azimuth of *BC* is 90° and that of *BA* is $30° + 180° = 210°$; hence, the required angle is $210° - 90° = 120°$.

17. Azimuth of Line From Azimuth of Another Line and Angle Between Lines.—Frequently, the angle between a line of known azimuth and a second line is measured, and it is required to compute the azimuth of that second line. For example, suppose that in Fig. 14 (*a*) the azimuth of AB is 61° and the angle a is 224°; let it be required to find the azimuth of AC. Evidently, the angle NAC clockwise, which is the azimuth of AC, is equal to $61°+224°=285°$. Now suppose that the azimuth of AD in (*b*) is 327° and the angle a is 160°. The angle NAD is $360°-327°$, and the angle NAE is $160°-(360°-327°)$, or $160°-360°+327°$, which may be written as $327°+160°-360°$.

Let B = azimuth of given line;
C = angle between that line and another, measured clockwise;
F = azimuth of second line.

Then from the foregoing explanations,

$$F=B+C \qquad (1)$$

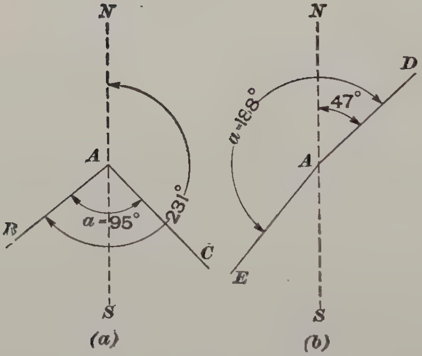


FIG. 15

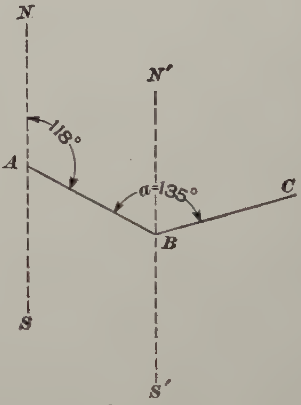


FIG. 16

In case the value of F, obtained by formula **1**, exceeds 360°, it is decreased by 360° in order to make the azimuth less than 360°.

When the angle from a given line to another line is measured counter-clockwise, the formula is

$$F=B-C \qquad (2)$$

in which F and B have the same meanings as in formula **1**, and C is the angle between the lines measured counter-clockwise. In case B is less than C, 360° is added to the value of B before the angle C is subtracted. If, in Fig. 15 (a), the azimuth of AB is 231° and the angle a is 95°, then the azimuth of AC may be found by formula **2**; thus,

$$F = B - C = 231^\circ - 95^\circ = 136^\circ$$

Now suppose that in (b) the azimuth of AD is 47° and the angle a is 188°. The azimuth of AE is, in this case, $47^\circ + 360^\circ - 188^\circ = 219^\circ$.

When the azimuth of AB, Fig. 16, and the angle a are given, the angle is added to the azimuth of BA (not AB) in order to determine the azimuth of BC. For example, suppose that the azimuth of AB is 118° and the angle a is 135°; then the azimuth of BA is $118^\circ + 180^\circ = 298^\circ$, and the azimuth of BC is $298^\circ + 135^\circ - 360^\circ = 73^\circ$.

18. True and Magnetic Azimuths.—The relations between the true and magnetic azimuths of a line are given by the following formulas:

Let T = true azimuth;
M = magnetic azimuth;
D = magnetic declination.

Then, if the declination is east,

$$T = M + D \qquad (1)$$

and

$$M = T - D \qquad (2)$$

If the declination is west,

$$T = M - D \qquad (3)$$

and

$$M = T + D \qquad (4)$$

EXAMPLE 1.—If the true azimuth of a line is 274° 28′ and the magnetic declination is 2° 40′ west, what is the magnetic azimuth of the line?

SOLUTION.—Since the declination is west, formula **4** applies; hence, the magnetic azimuth of the line is $274^\circ\ 28' + 2^\circ\ 40' = 277^\circ\ 8'$. Ans.

EXAMPLE 2.—The magnetic azimuth of a line is 48° 15′ and the magnetic declination is 3° 15′ east. Find the true back azimuth of the line.

SOLUTION.—Since the declination is east, the true forward azimuth of the line is found by formula **1**; thus,

$$T = M + D = 48^\circ\ 15' + 3^\circ\ 15' = 51^\circ\ 30'$$

The required back azimuth is $51^\circ\ 30' + 180^\circ = 231^\circ\ 30'$. Ans.

19. Azimuths and Bearings.—A line in a given direction has but one bearing and one azimuth. The bearing can be found from the azimuth, or vice versa, by one of the following rules:

Rule I.—*If the azimuth is between 0° and 90°, the bearing is northeast and is equal to the azimuth; conversely, if the bearing is northeast, the azimuth is equal to the bearing.*

Rule II.—*If the azimuth is between 90° and 180°, the bearing is southeast and is equal to 180° minus the azimuth; conversely, if the bearing is southeast, the azimuth is equal to 180° minus the bearing.*

Rule III.—*If the azimuth is between 180° and 270°, the bearing is southwest and is equal to the azimuth minus 180°; conversely, if the bearing is southwest, the azimuth is equal to 180° plus the bearing.*

Rule IV.—*If the azimuth is between 270° and 360°, the bearing is northwest and is equal to 360° minus the azimuth; conversely, if the bearing is northwest, the azimuth is equal to 360° minus the bearing.*

Example 1.—Find the bearings corresponding to the following azimuths: (*a*) 62° 10′; (*b*) 111° 31′; (*c*) 200° 14′; and (*d*) 348° 48′.

Solution.—(*a*) By rule I, the bearing is N 62° 10′ E. Ans.
(*b*) By rule II, the bearing is S (180° − 111° 31′) E = S 68° 29′ E. Ans.
(*c*) By rule III, the bearing is S (200° 14′ − 180°) W = S 20° 14′ W. Ans.
(*d*) By rule IV, the bearing is N (360° − 348° 48′) W = N 11° 12′ W. Ans.

Example 2.—Find the azimuths of the lines whose bearings are: (*a*) N 16° 32′ E; (*b*) S 16° 32′ E; (*c*) S 16° 32′ W; (*d*) N 16° 32′ W.

Solution.—(*a*) By rule I, the azimuth is 16° 32′. Ans.
(*b*) By rule II, the azimuth is 180° − 16° 32′ = 163° 28′. Ans.
(*c*) By rule III, the azimuth is 180° + 16° 32′ = 196° 32′. Ans.
(*d*) By rule IV, the azimuth is 360° − 16° 32′ = 343° 28′. Ans.

EXAMPLES FOR PRACTICE

1. Find the true back azimuth of a line whose magnetic forward azimuth is 116° 17′, the magnetic declination being 1° 30′ west.
Ans. 294° 47′

2. If the true azimuth of a line is 249° 21′ and the magnetic declination is 2° 20′ east, what is the magnetic back azimuth of the line? Ans. 67° 1′

3. Find the bearings corresponding to the following **azimuths**: (*a*) 94° 12′; (*b*) 358° 30′; (*c*) 3° 14′; (*d*) 269° 47′.

Ans. (*a*) S 85° 48′ E; (*b*) N 1° 30′ W;
(*c*) N 3° 14′ E; (*d*) S 89° 47′ W

4. Find the azimuths of the lines having the following bearings: (*a*) N 88° 16′ W; (*b*) S 5° 7′ E; (*c*) S 5° 7′ W; (*d*) N 88° 16′ E.

Ans. (*a*) 271° 44′; (*b*) 174° 53′;
(*c*) 185° 7′; (*d*) 88° 16′

5. If the azimuth of a line *OA* is 130°, that of *OB* is 250°, and that of *OC* is 307°, find the values of the following angles: (*a*) *AOB*, clockwise; (*b*) *AOC*, counter-clockwise; (*c*) *COB*, counter-clockwise, (*d*) *BOA*, clockwise.

Suggestion.—Make a sketch and show the azimuths of the lines as in Fig. 12.

Ans. (*a*) 120°; (*b*) 183°;
(*c*) 57°; (*d*) 240°

6. The azimuth of a line *AB* is 110°, and that of *BC* is 175°; find the angle *ABC*, measured clockwise. Make a sketch similar to Fig. 13.

Ans. 245°

7. The azimuth of a line *OA* is 130°. (*a*) If the angle *AOB*, clockwise, is 176°, find the azimuth of *OB*. (*b*) If the angle *AOC*, counter-clockwise, is 235°, find the azimuth of *OC*.

Ans. (*a*) 306°
(*b*) 255°

8. If the azimuth of a line *AB* is 45° and the angle *ABC*, clockwise, is 200°, what is the azimuth of *BC*? Ans. 65°

OPERATIONS WITH TRANSIT

GENERAL EXPLANATIONS

20. Transit Points.—A point over which the transit is set up is called a *transit point*. Such a point should be marked as accurately as possible on some firm object. In surface surveys, a firmly embedded rock, or a specially prepared concrete block, called a *monument*, is preferable for locating the most important points; elsewhere, wooden stakes, called *hubs*, are driven flush with the ground. The point is marked on rock by a chiseled cross; in concrete a mark is made while the concrete is soft, or a tack, a nail, or a small pipe is embedded in the soft concrete; the point on a hub is marked by a flat-headed tack, flush with the top of the hub. For identifying points on rock or concrete the necessary facts can be written on the stone with *keel*. To identify a hub and to indicate its location, a projecting stake, which is called a *guard stake* or *witness stake*, is placed near the hub; on it are marked in keel the station number of the hub and any other necessary information, such as the name or the purpose of the survey.

21. Motions of Telescope.—As previously explained, the telescope of a transit has two separate turning motions; namely, rotation about the transverse axis and rotation about the axis of the instrument.

When the telescope is in the position shown in Fig. 1, that is, when the level is under the telescope, it is said to be *normal*. However, it is frequently convenient to turn the telescope on the transverse axis so that it points in the opposite direction and the level is above the telescope. The telescope is then said to be *plunged* or *reversed*. The operation of turning the telescope from its normal to its reversed position, or vice versa, is called *plunging the telescope*. The telescope can be directed toward a point when in either position. For reference, the vernier near the eyepiece when the telescope is normal is marked *A* and the other is marked *B*.

When the instrument is rotated on its vertical axis, it is said to be *rotated in azimuth* because the azimuth of the line of sight changes during the turning. This motion can be effected either by rotating the upper plate alone or by revolving both plates together. In the first case, the lower plate is clamped and the upper plate is unclamped; then the verniers move around the horizontal limb. In the second case, the lower plate is unclamped and the upper plate is clamped; in this motion, the verniers remain fixed with respect to the limb. The operation of turning the instrument through exactly one-half of a revolution, or 180°, is called *reversing in azimuth*. The operations of plunging the telescope and reversing in azimuth both have the effect of pointing the telescope in the opposite direction.

22. Setting Up.—In setting up a transit, it is important to have the center of the instrument, which is the point of intersection of the transverse axis and the vertical axis of the instrument, directly over a given point on the ground; also, the plates and the transverse axis must be horizontal. Much of the work of a surveying party is suspended while the transit is being set up; speed in setting up is, therefore, very desirable.

As previously explained, the location of the center of the transit is indicated by a plumb-bob, suspended from a hook or ring attached to the instrument at q', Fig. 3. The plumb-bob string should be held by a sliding knot in order that the height of the bob can be adjusted; the point of the bob should almost touch the mark over which it is desired to set the transit.

In setting up, the tripod legs are spread, and their points are so placed that the leveling head is approximately horizontal and the telescope is at a convenient height for sighting; the instrument should be within a foot of the desired point but no extra care should be taken to set it very close at once. For convenience in setting up on the side of a hill and over points in rough ground, the legs of the tripod are often made telescopic, in order that their lengths can be adjusted; in setting up on a slope, two legs should be down hill. If the instrument is more than a few inches from the given point, the tripod is lifted without changing the inclinations of the legs and the instrument set as near as possible to the point. By pressing the legs firmly into the ground, the plumb-bob can usually be brought to within a quarter of an inch of the point. If the leveling head is tipped too much, two or, if necessary, all three legs of the tripod are moved to new positions and the plumb-bob is again brought close to the point by pressing the legs into the ground.

When the leveling head is approximately horizontal and the plumb-bob is very near the point, the bubbles of the plate levels are brought almost to the centers of the tubes by means of the leveling screws. If there are four screws, the plates are rotated until one level is parallel to each pair of opposite screws; then each bubble is brought to the center separately by turning the screws of the corresponding pair as explained for the wye level. If there are three screws, the plates are rotated so that one level is parallel to any two screws; both bubbles are then brought to the center at the same time by turning one screw of that pair and the third screw.

After the bubbles are approximately centered, the instrument is loosened on the tripod plate by loosening two adjacent leveling screws, and the plumb-bob is brought exactly over the point on the ground by means of the shifting head. Then the instrument is held securely in position on the tripod plate by tightening the screws that were previously loosened. The bubbles of the plate levels are brought exactly to the centers of the tubes, and the position of the plumb-bob over the point is observed. If the bob has moved off the point in leveling up, it is brought back to the proper position by means of the shifting head, and the instrument leveled again.

23. Review of Functions of Clamps and Tangent Screws. The manner in which the clamps and tangent screws work has already been explained. The subject, however, is here restated in a different form, as a thorough understanding of it is of the greatest importance for an intelligent handling of the transit.

The lower clamp and the lower tangent screw control the motion of the lower plate about the vertical axis. When the lower clamp is set, the lower plate, and, therefore, the instrument as a whole, cannot be revolved in azimuth except very slowly by means of the lower tangent screw. The upper clamp and tangent screw control the sliding motion of the upper plate over the lower. When the upper clamp is set, the upper plate cannot be revolved over the lower except very slowly by means of the upper tangent screw. When the upper clamp is set and the lower loosened, the reading of the instrument is not altered by rotating the telescope. When the lower clamp is set and the upper loosened, the vernier slides along the graduated circle and the reading of the vernier is changed.

24. Directing Telescope to Given Mark.—The telescope, or the line of sight, is said to be directed to a given point when, both plates being clamped, the vertical cross-wire of the transit passes through the point. To apply the method of performing this operation, suppose that, the instrument being set up and leveled at some point, it is desired to direct the line of sight toward a certain mark, as a pole held at one of the stations of a survey.

First one clamp (but not both) must be loosened. If the reading of the vernier is to remain unchanged, the lower clamp should be loosened; otherwise, the upper clamp. The instrument is then revolved in azimuth, one hand being placed near the bottom of each standard. The transitman, looking over the telescope, points it toward the flagpole to be observed, as nearly as he can estimate by the eye. He then looks through the telescope, and, if necessary, turns it to one side or the other, and up or down, until the pole appears in the field of view. Still turning the plate with his hands, he brings the image of the flagpole nearly into coincidence with the vertical cross-wire, as shown in Fig. 17; here $a\,b$ and $c\,d$ are the cross-wires, and PQ is the image of the flagpole. He now sets the clamp that has been loosened, and turns the corresponding tangent screw until the intersection of the cross-wires exactly bisects the pole, as shown in Fig. 18. This completes the operation.

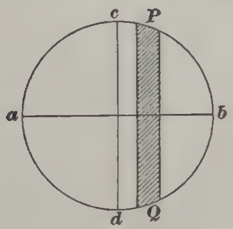


FIG. 17

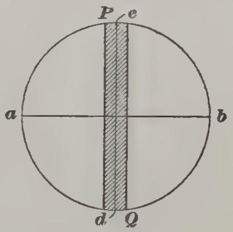


FIG. 18

In transit surveying, the point sighted at should be well defined. The point of a pencil held in the proper position, or the string of a plumb-bob suspended over the point, is preferable for sighting; however, if the sight is very long and such objects cannot be seen distinctly, a range pole can be used. The sight is taken in the same way whether the telescope is normal or reversed. In using a tangent screw, it is important to make sure that the last turn of the screw tightens it against the opposing spring; if the last turn loosens the screw, the spring may not press against the projecting piece between the screw and the spring, and thus the telescope may move after it is set or may not be set properly.

FIELD PROBLEMS

25. Prolonging a Line.—Let AB, Fig. 19, be a line whose position on the ground is fixed by hubs at A and B. The line can be prolonged to C in two ways. In one method, the transit is set up at A, and a sight is taken to B. The point C is located by setting a hub in the line of sight, and marking a point on the hub by means of a tack. If the distance from B to C is to be chained, the chainmen can be lined in by the transitman.

FIG. 19

This method of prolonging a line is often impracticable, since the point C is not always visible from A.

A simple method sometimes used in practice is to set the transit at B, sight to A, and then plunge the telescope so that it will point in the opposite direction, that is, along AB prolonged, or BC. When greater accuracy is required, or when there is reason to believe that the transit is not in adjustment, one plate is unclamped, and, with the telescope still reversed, another sight is taken to A. Then the telescope is plunged back to its normal position, and the new direction for AB prolonged is compared with that previously obtained. If the two positions of C do not coincide, the correct point is located midway between them. The process of prolonging a line by taking the average of two sets of sights as just described is called *double centering*.

Just as in the case of turning an angle, the sight along BA is called a backsight and the sight along BC is a foresight. Sometimes, the point A is also referred to as the backsight, and the point C is called the foresight.

26. Measuring Horizontal Angles.—In Fig. 20, let MO and ON be two lines on the ground. To measure the angle

MON, place the transit at *O* and set the index of a vernier, say *A*, to read zero on the horizontal circle by the following method. Unclamp the upper plate and rotate it with respect to the lower plate until the index is within a half degree of the zero of the limb. Then clamp the upper plate and, by means of the upper tangent screw, bring the index exactly opposite the zero of the limb.

With the index clamped at zero and the lower plate unclamped, bring the vertical cross-wire nearly on the point *M* and clamp the lower plate; then bring the intersection of the cross-wires exactly on *M* by means of the lower tangent screw. Next, unclamp the upper plate and bring the vertical wire nearly on the point *N*; clamp the upper plate and bring the intersection of the wires exactly on *N* by means of the upper tangent screw. Then the reading of the horizontal circle at the index of vernier *A* is the value of the angle *MON*. In Fig. 20, the angle is shown as 143° 30′ clockwise, which would be the reading indicated by the inner set of numbers on the limbs shown in Fig. 9. Care must be taken to use the proper tangent screw in each case, as turning the wrong screw is a common source of error.

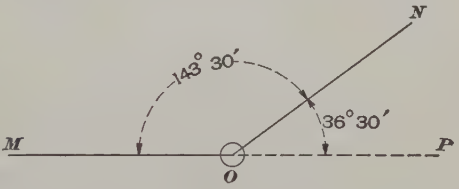


FIG. 20

It is not always necessary to set the index at zero before sighting along the backsight. When the horizontal limb is numbered continuously from 0 to 360 it is sometimes more convenient to read the vernier, wherever it may be, when the instrument is sighted on *M* and both plates are clamped. Then the upper plate is unclamped and the reading for the sight to *N* is taken. The difference between the two readings, taken with the same vernier and on the same set of graduation numbers, is the value of the angle. For instance, the reading for the sight to *M* may have been 140° and that for the sight to *N*, 283° 30′; then the angle *MON* is 283° 30′ − 140° = 143° 30′. The method explained in the preceding paragraph is usually preferable, however, as there is less chance of confusion and error.

27. Deflections.—The relative directions of two lines may be given by the angle between one line and the prolongation of the other line. Thus, in Fig. 21 (*a*), instead of taking the angle AOB, it is often more convenient to consider the angle $A'OB$ between OB and the prolongation of AO. The angle that one line makes with another line produced is called the deflection of the first line from the second. Deflections are measured either clockwise or counter-clockwise from the prolongation of the backsight. Thus, in Fig. 21 (*a*), the deflection of OB from AO is 20° clockwise, usually designated 20° to the right and written 20° R. In (*b*), the deflection of OC from AO is 145° to the left, or 145° L. Deflections are always measured in the direction that gives an angle less than 180°.

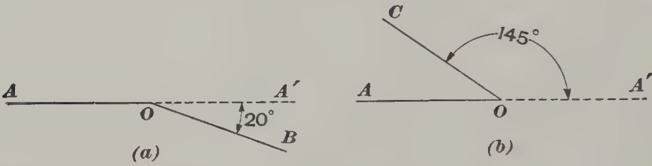


Fig. 21

28. Measuring Deflection Angles.—To measure the deflection of ON from MO in Fig. 20, the transit is set up at O, the index of a vernier, say A, is set at 180° of the horizontal circle, and, with the telescope normal, the intersection of the cross-wires is brought exactly on the point M by means of the lower clamp and its tangent screw. Then the upper plate is unclamped, and the line of sight is directed to N by means of the upper clamp and its tangent screw. The reading of the vernier A is the value of the angle $P\,O\,N$, which is the required deflection angle, because it is the supplement of the angle $M\,O\,N$, or $180° - M\,O\,N$. In Fig. 20, the angle $P\,O\,N$ is given as 36° 30′; since it is turned to the left from $O\,P$, it would be recorded as 36° 30′ L.

29. Passing Obstacles.—The methods of passing obstacles with a transit are similar to those described for a compass; but, instead of using the bearings of the lines, the angles between the lines are taken. For example, suppose that it is required to determine the length of the line AB, Fig. 22, which crosses a river. First, the line is run to a point P near the bank of the river and the transit is set up at that point. Then any convenient line PQ is laid off, and the angle P and the distance PQ are measured. Finally, the transit is set up at Q and the angle Q is measured. The angle B is equal to 180° $-P-Q$; but, if possible, the angle B should be measured in order that the work may be checked by seeing whether the sum of the three angles is 180°. The length of PB is computed by the relation $PB = \frac{PQ \sin Q}{\sin B}$, and the length of AB is found by adding PB and AP.

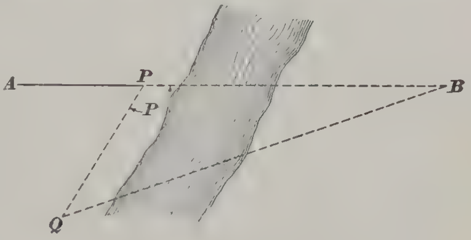


FIG. 22

SURVEYING WITH TRANSIT

RUNNING TRANSIT LINES

INTRODUCTION

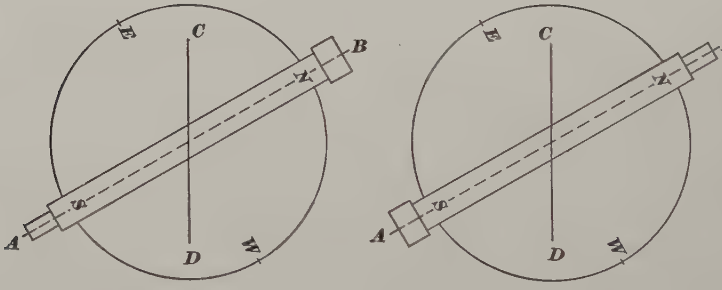
30. Measuring Distances.—Distances in a transit survey should be accurate and, therefore, a steel tape should be used for measurements by chaining. The tape should be held straight, horizontal, and taut; and when the position of a point is to be transferred from the tape to the ground, or vice versa, a plumb-bob should be employed.

As already explained, distances should be measured horizontally, but it is sometimes more convenient to measure a distance parallel to the ground and the angle that the inclined line makes with the horizontal. The required horizontal distance may then be computed by multiplying the inclined distance by the cosine of the vertical angle. Occasionally, the difference in elevation between the ends of the line is determined instead of the angle of inclination. In this case, the horizontal distance is found by the following relation between the sides of a right triangle.

Let a = horizontal distance;
b = inclined measurement;
c = difference in elevation between ends.

Then,
$$a=\sqrt{b^2-c^2}$$

31. Determining Directions.—There are three common methods of determining the directions of the courses of a traverse by means of a transit: (1) by direct angles; (2) by deflection angles; and (3) by azimuths. Each method has advantages for certain classes of surveys.



(a) (b)

Fig. 23

Most transits have a needle and a needle circle and can, therefore, be used as a compass, the telescope being substituted for the sights. In conjunction with any of the foregoing methods, the bearings of some of the courses in a transit survey are commonly observed on the needle circle as an approximate check to guard against mistakes in reading the limb of the transit. In such cases, the bearings are usually read to the nearest quarter degree.

When a sight is taken with the telescope reversed, the bearing of the line is indicated by the south end of the needle. The following explanation gives the reason for this. Suppose, for example, that the line of sight is directed along the line AB, Fig. 23 (*a*), with the telescope normal; in this case, the eyepiece is toward A and over the south point S of the needle circle. The needle CD (the north end is at C) indicates that the bearing of AB is northeast, the north end being read when the telescope is normal. Now, suppose that the telescope is plunged to its reversed position. The conditions are then shown in (*b*); the eyepiece is toward B and the line of sight is directed along BA instead of AB. Since the needle circle has not been rotated, the reading of the needle is not altered. Hence, the north end of the needle still indicates a northeast bearing although the line of sight is now directed southwest. The south end of the needle, on the other hand, properly records a southwest bearing.

METHOD OF DIRECT ANGLES

32. General Description.—In the method of determining the directions of the courses of a transit survey by direct angles, the transit is set up at each corner; and each angle is measured by taking a backsight along one course and a foresight along the next course. The angles may be measured in any order but, to avoid confusion, all angles of the survey should be turned in the same direction, preferably clockwise.

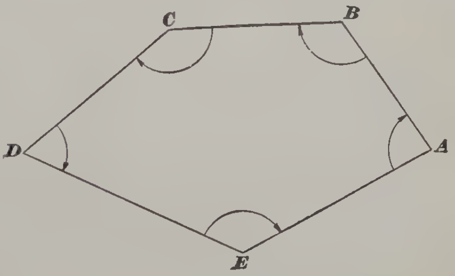


Fig. 24

To survey the boundaries of the field shown in Fig. 24, the transit is set up at any corner, say A, a backsight is taken on E, and the angle EAB is measured clockwise, as indicated by the arrow; the value is recorded in the notes. The magnetic bearing of AB should be taken by reading the needle circle, and the distance AB should be measured. Then the transit is moved to B, and a backsight is taken along BA. The angle

ABC is measured clockwise and the length and the magnetic bearing of *BC* are determined. The other angles, lengths, and bearings are determined in a similar manner.

The magnetic bearings are used as a check on the angles. Suppose, for instance, that the angle at *B*, Fig. 24, is 117° 42′, and the bearings of *AB* and *BC* are N 37° 30′ W and S 80° 15′ W, respectively. Then the bearing of *BC* is computed from the observed values of the bearing of *AB* and the angle at *B*, as follows: The conditions are shown in Fig. 25; *NS* represents the meridian through *B*, the angle *SBA* is 37° 30′, and the angle *ABC* is 117° 42′. Then *SBC* is equal to $ABC - SBA = 117° 42' - 37° 30' = 80° 12'$, and the bearing of *BC* is S 80° 12′ W. Since this agrees closely with the observed bearing of *BC*, the angle is taken as correct. In case the observed and calculated values of a bearing do not agree within ½ degree, all readings and computations connected with that bearing should be verified. If no error is found, the difference is probably due to local attraction.

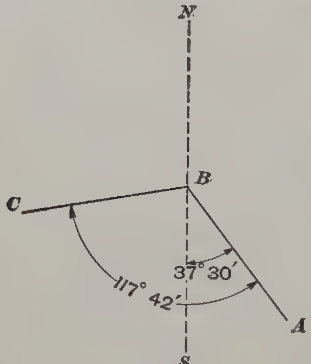


FIG. 25

33. Field Notes.—There are many forms of keeping the field notes for the method of surveying by direct angles, but only one will be shown here to serve as a guide. The title of the survey, the date, the names of the members of the party, and other pertinent information should be given as explained for a compass survey and as shown in Fig. 26. In the first column of the left-hand page of the notebook (a transit book is used) is the angle, the first letter indicating the point on the backsight, the second letter denoting the vertex, and the third letter the point on the foresight; a note to this effect may be inserted at the bottom of the left-hand page to avoid confusion. The second column contains the value of the angle measured clockwise, as indicated by the heading. In the third column is the course, for which the length is given in the fourth column

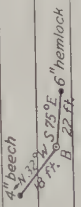
16 Angle	Value Clockwise	Course	Length	Magnetic Bearing Obs.	 Cal.	Survey of Brown Farm 17 'Elmhurst, Pa. July 27, 1925 M. Wood, Transit R. Jackson } Chainmen S. White
EAB	106°41'	AB	361.43	N $30\frac{1}{4}$° W	N 30°12'W	A is concrete monument. True bearing of AB found by observation of true
						meridian to be N 28°16'W. B is large hub referenced as in sketch.
ABC	153°17'	BC	506.85	N 57° W	N 56°55'W	C is hub at edge of road.
						 4" beech N 32° W 8 ft. S 75° E 22 ft. B 6" hemlock
BCD	129°32'	CD	483.79	S $72\frac{1}{2}$° W	S 72°37'W	

FIG. 26

and the observed and magnetic bearings in the last two columns of the left-hand page. The notes may be read either from the top of the page downwards or from the bottom of the page upwards.

A complete description of each corner, with references to permanent objects near-by, and any necessary remarks and sketches, should be given on the right-hand page of the notebook. The true bearing of some line of the survey should be determined either from an astronomical observation on the true meridian or from a line of another survey that has been previously established.

34. Measuring Angles by Repetition.—The measurement of direct angles is best adapted to closed traverses where extreme accuracy is required, because the angles can be determined by the following method, called the *method of repetition*, with greater precision than the least reading of the vernier. In this process, the vernier is set at zero and the telescope is directed to the backsight; for instance, if the transit is set up at *A* in Fig. 24, the telescope is sighted to E. Then the upper plate is unclamped and the telescope is sighted on the foresight, B. The vernier reading is recorded for reference. With the vernier still set in this position, the lower plate is unclamped and the telescope is again directed to E as a backsight. Then, the upper plate is unclamped and another foresight is taken to B. The vernier should now read twice the angle EAB, but it is not necessary to observe the reading. With the vernier clamped at this reading, the lower plate is again unclamped and a third backsight taken on E. Then the upper plate is unclamped and a foresight is taken to B. Now the vernier reads three times the angle $E\,A\,B$. The operation may be repeated as often as desired, the value of the angle being obtained by dividing the final reading by the number of times the angle was turned. For example, suppose the transit can be read to 30 seconds and the first reading is 26° 40′ 30″. Then, after the angle has been turned four times, suppose the vernier reads 106° 41′ 30″. The average value of the angle, which is taken as the correct value, is, therefore, equal to $\frac{1}{4}\times 106° 41' 30'' = 26° 40' 22.5''$. Again, suppose that an angle is found to be about 105° 21′ and that the reading after the angle has been turned six times is 272° 7′. Since the angle is approximately 105° 21′, six times the angle should be about $6\times 105°\ 21'$, or 632° 6′; hence, the index of the vernier must have turned through more than a complete revolution with respect to the limb, and the final reading may be considered as $360° + 272°\ 7'$, or 632° 7′. The value of the angle, in this case, is taken as $\frac{1}{6}\times 632°\ 7'$, or 105° 21′ 10″.

By this method a different part of the limb is generally used in each measurement of the angle and the errors due to poor graduation of the limb are practically eliminated. Moreover the unavoidable errors in reading the vernier are not multiplied for each turning.

METHOD OF DEFLECTION ANGLES

35. Outline of Process.—In the method of deflection angles, the directions of the courses are determined by measuring the angle that each line makes with the prolongation of the preceding one. This method is most convenient when all the lines of the traverse have the same general direction, as is usually the case in a survey for a railroad.

To illustrate the process, suppose it is required to run a deflection traverse from the point *F*, Fig. 27, the true bearing of *F G* being known from a previous survey to be S 16° 35′ W. The transit is set up at *F*, and, with the vernier set at 180° and the telescope normal, a backsight is taken to *G*; the magnetic bearing of *F G* is observed by reading the compass needle, and is compared with the true bearing so that the magnetic declination can be determined. The upper plate is then unclamped, a sight is taken to *A*, and the reading of the vernier is recorded; since *F A* is to the right of *F G* produced, the reading is marked 44° 28′ R. The magnetic bearing of *F A* is also taken and the distance *F A* is measured. The transit is moved to *A*, and the deflection of *A B* from *F A* is measured and recorded as 57° 42′ R; the bearing and length of *A B* are also determined, and the observed bearing is compared with the value calculated from the bearing of *F A* and the deflection of *A B*. The deflections, bearings, and distances of the other courses are obtained in a similar manner.

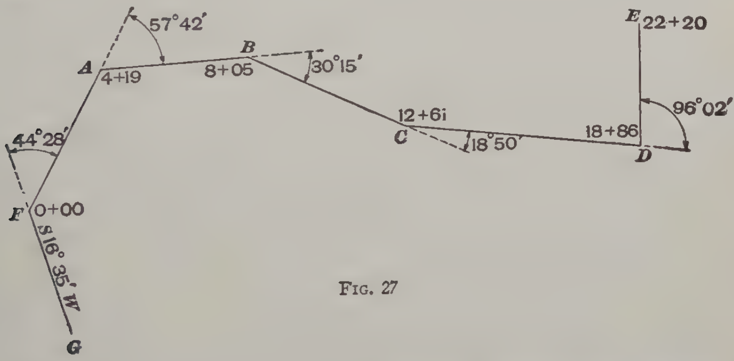


FIG. 27

In a railroad survey, the point *F* would be called Station 0+00 because it is the beginning of the traverse, and the point *A* would be numbered to correspond to the distance *F A*; thus, if *F A* is 419 feet, *A* is station 4+19. The station numbers of the other points are found in a similar way; the distance from *F* to *B* is 419+386=805 feet, and the station number is, therefore, 8+05.

36. Field Notes.—A good method of recording the field notes for a deflection traverse like that shown in Fig. 27 is given in Fig. 28. The general information concerning the purpose of the survey, the members of the party, etc., should be given as in Fig. 26. In the first column of the left-hand page are the station numbers; in the second column, the distances; in the third, the deflections· and in the fourth and fifth, the observed and calculated magnetic bearings. The notes read upwards from the bottom of the page and the values on the line of the notes between any two stations refer to the course in the field joining the same two stations. Thus, 57° 42′ R, between Stations 4+19 and 8+05 in the notes, is the deflection of the course from Station 4+19 to Station 8+05; similarly, 625 in the notes between Stations 12+61 and 18+86 is the distance from Station 12+61 to Station 18+86.

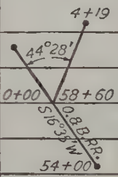
30							31
Sta.	Dist.	Deflect.	Magnetic Bearing			Remarks	
			Obs.	Cal.			
22+20						End of Line	
	334	96°02′L	N 32½°E	N 32°33′E			Sta. 0 is at Sta. 58 + 60
18+86							of O. and B.RR. Backsight on
	625	18°50′L	S. 51½°E	S 51°25′E			Sta. 54 + 00 of same line
12+61							True bearing S 16°35′W
	456	30°15′R	S 32½°E	S 32°35′E			and magnetic bearing S 15°0′W.
8+05							
	386	57°42′R	S 62¾°E	S 62°50′E			
4+19							
	419	44°28′R	N 59½°E	N 59°28′E			
0+00							

FIG. 28

The column headed *Remarks* represents the entire right-hand page of the notebook. The reference line from which the survey is started should be fully described, and the relation between the first line of the new survey and the old established line should be given by a sketch as shown. Often the deflections to the right and those to the left are placed in separate columns, headed *right* and *left*, respectively.

METHOD OF AZIMUTHS

37. Orienting.—As previously explained, the azimuth of a line is the angle between the line and a meridian. Although in a survey by azimuths the true meridian is always preferable and sometimes necessary, the magnetic meridian or any other convenient line, called a *reference meridian*, is often used as the direction from which azimuths are measured. The important consideration is that all azimuths of a traverse must be referred to the same meridian, which should be fully described.

When the telescope is directed along a given line and the reading of the vernier of the transit indicates the azimuth of the line of sight, the transit is said to be *oriented*. For example, if the telescope is in the magnetic meridian and the vernier reads zero, the transit is oriented for magnetic azimuths. Likewise, if the true azimuth of a certain line is known to be 30°, and the vernier reads 30° when the telescope is directed along the line, the transit is oriented for true azimuths.

38. Methods of Orienting.—In starting a survey, there must be some line of which the azimuth is either known or assumed and along which the transit can be oriented. There are three methods of orienting, all of which are similar in principle.

One method is based on the fact that the back azimuth of a line is equal to its forward azimuth plus or minus 180°. For example, suppose that the azimuth of OP, Fig. 29, is known to be $64° 21'$, and the transit is set up at P. Then the azimuth of PO, which is the back azimuth of OP, is $64° 21' + 180°$, or $244° 21'$. Hence, if vernier A is set to read $244° 21'$ and the telescope is directed to O, the transit will be oriented along the line PO.

As previously explained, the two verniers on a transit are exactly 180° apart. A second method of orienting utilizes the fact that when vernier *B* reads the azimuth of *OP*, vernier *A* reads that value plus or minus 180°, which is the azimuth of *PO*. Hence, if vernier *B* is set at the azimuth of *OP* and the telescope is directed to *O*, the transit will be oriented for reading vernier *A*.

The third method is based on the principle of prolonging a line by plunging the telescope. If a backsight is taken to *O* with the telescope reversed and then the telescope is plunged back to its normal position, the line of sight will be directed along *PQ*, or *OP* produced, and the azimuth of the line of sight will be equal to the azimuth of *OP*. Hence, in this method of orienting, vernier *A* is set to read the azimuth of *OP*, a backsight is taken to *O* with the telescope reversed, and then the telescope is plunged back to normal. This third method is the most convenient, but is not so accurate unless the transit is in perfect adjustment.

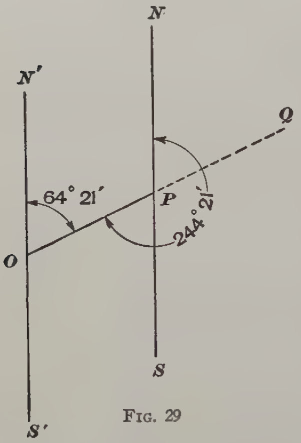


FIG. 29

Often a traverse is started by orienting the transit in the magnetic meridian as follows: Set up over the starting point and loosen the needle. Set the vernier to read zero and then rotate the instrument until the north end of the needle is exactly opposite the north point of the needle circle. Since the vernier reads zero and the telescope is in the magnetic meridian, the transit is oriented.

39. Azimuth Traverse.—Suppose that the traverse shown in Fig. 30 starts at *A*, which is a known point on the line *AF*, whose true bearing was found in a previous survey to be N 42° 36′ W. Since the bearing of *AF* is northwest, its azimuth is 360° − 42° 36′, or 317° 24′. The transit is set up at *A* and is oriented by sighting to *F* with the vernier reading the azimuth of *A F*, or 317° 24′; the magnetic bearing of *A F* is observed and recorded. The upper plate is then loosened and the azimuth of the line of sight for any position of the telescope is given by the reading of the vernier. Thus, if the telescope is directed to *B*, the reading of the vernier, in this case 75° 17′, is recorded as the azimuth of *A B*. The distance from *A* to *B* is measured and recorded, and the bearing of the line is observed.

In using the third method of orienting the transit, the vernier is left set at the azimuth of the preceding course. However, the vernier should be observed before backsighting to make sure that the setting has not changed while the transit was being carried ahead. Thus, when the transit is moved to *B*, the vernier is left at the azimuth of *A B*, and the backsight to *A* is taken with the telescope reversed. Then the telescope is plunged back to normal, the upper plate is unclamped, and the telescope is directed to *C*. The reading of the vernier gives the azimuth of *B C*, in this case 89° 30′; the bearing is also observed as a check, and the distance *B C* is measured. The vernier is left at the azimuth of *B C*; the transit is then set up at *C* and is oriented by backsighting to *B* with the telescope reversed. The operations just described are repeated for each point.

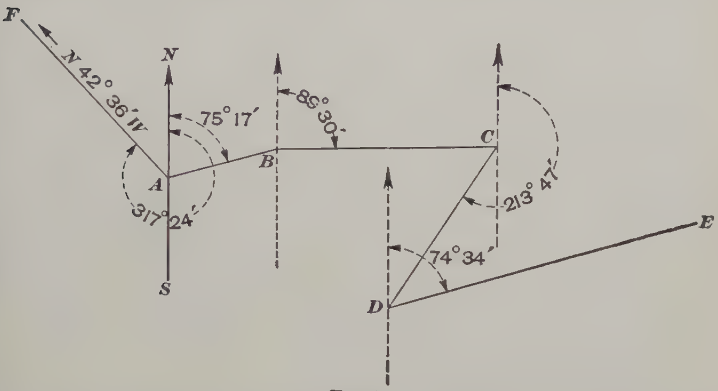
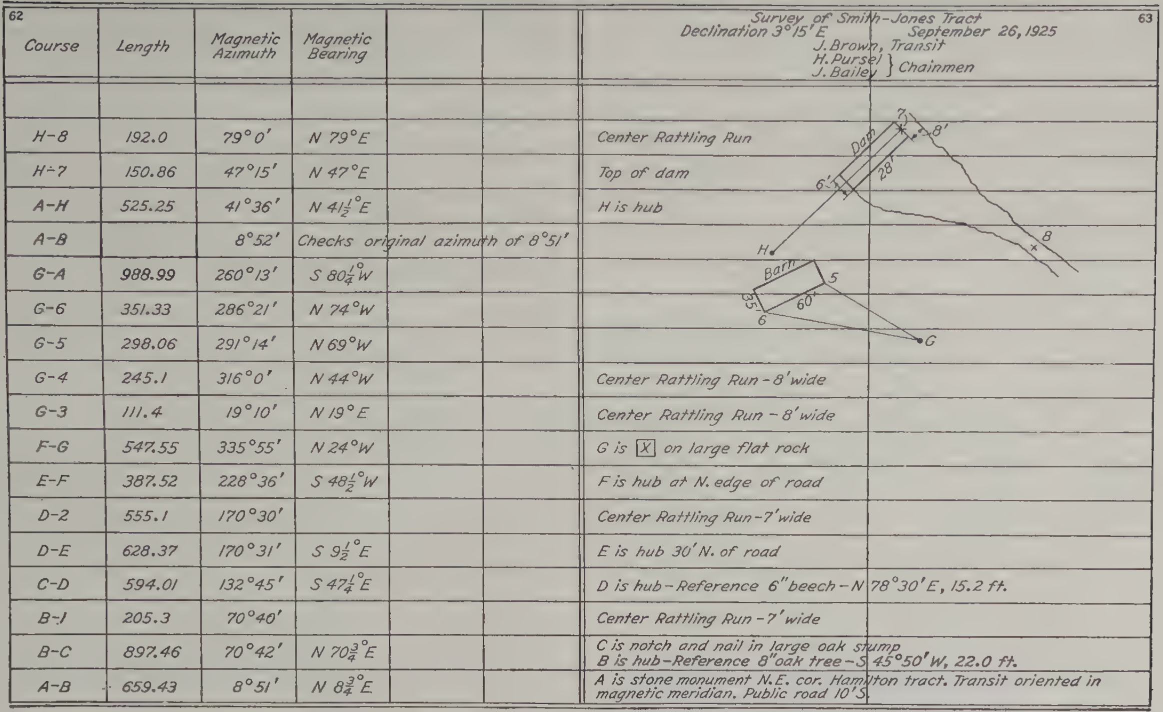


FIG. 30

40. In azimuth surveying, it is convenient to refer the bearings to the same meridian from which the azimuths are measured in order that the bearing and the azimuth of each course may be readily compared in the field. To convert true azimuths to magnetic, or vice versa, the magnetic declination must be known. If it is not given for the locality, it may be found by taking the difference between the true and magnetic bearings of any line. For instance, if the magnetic bearing of AF, Fig. 30, is read on the needle circle as N 40° 15′ W, the angle between the true and magnetic meridians at A is $42° 36' - 40° 15' = 2° 21'$, say 2° 20′. Since the north point of the needle lies to the west of the true meridian, the declination may be taken as 2° 20′ W. If it is suspected that there is local attraction at A, the true and magnetic bearings of some other line must be used.

In most cases where azimuths are used, magnetic azimuths are determined in the field, and the lines are drawn on a map and marked with respect to the magnetic meridian. Then the true meridian is drawn on the map and the angle it makes with the magnetic meridian is indicated; hence, true azimuths can be readily determined. If it is desired to measure true azimuths in the field, true bearings can also be read directly by setting the declination arc that is supplied on most transits.

41. Advantages of Azimuths.—The method of azimuths is best adapted to surveys where many points are to be located from a single set-up as in the case of a topographic survey. One advantage of azimuths is that all angles in the notes are measured in the same direction from the meridian as a backsight. Hence, the chance of an error in recording or interpreting the notes is much less than in recording direct or deflection angles, which must be marked left or right and for which it is necessary to indicate the backsight. Another advantage is that azimuths and bearings can be readily compared in the field; large errors in the transit work can thus be detected at once. In traversing by direct or deflection angles, the work must be delayed while the bearing is calculated from the bearing of the preceding course and the angle, or the calculations must be performed in the office, in which case an error may nullify much field work. A third advantage is that in office calculations, the azimuths or bearings of the courses are usually



62

63

Survey of Smith-Jones Tract
Declination 3°15′E
September 26, 1925
J. Brown, Transit
H. Pursel }
J. Bailey } Chainmen

Course	Length	Magnetic Azimuth	Magnetic Bearing			
H-8	192.0	79°0′	N 79°E			Center Rattling Run
H-7	150.86	47°15′	N 47°E			Top of dam
A-H	525.25	41°36′	N 41½°E			H is hub
A-B		8°52′	Checks original azimuth of 8°51′			
G-A	988.99	260°13′	S 80¼°W			
G-6	351.33	286°21′	N 74°W			
G-5	298.06	291°14′	N 69°W			
G-4	245.1	316°0′	N 44°W			Center Rattling Run - 8′ wide
G-3	111.4	19°10′	N 19°E			Center Rattling Run - 8′ wide
F-G	547.55	335°55′	N 24°W			G is ☒ on large flat rock
E-F	387.52	228°36′	S 48½°W			F is hub at N. edge of road
D-2	555.1	170°30′				Center Rattling Run - 7′ wide
D-E	628.37	170°31′	S 9½°E			E is hub 30′ N. of road
C-D	594.01	132°45′	S 47¼°E			D is hub - Reference 6″ beech - N 78°30′E, 15.2 ft.
B-1	205.3	70°40′				Center Rattling Run - 7′ wide
B-C	897.46	70°42′	N 70¾°E			C is notch and nail in large oak stump B is hub - Reference 8″ oak tree - S 45°50′W, 22.0 ft.
A-B	659.43	8°51′	N 8¾°E			A is stone monument N.E. cor. Hamilton tract. Transit oriented in magnetic meridian. Public road 10′ S.

FIG. 31

needed. The fourth important consideration is that plotting with a protractor is much more convenient if azimuths or bearings are used.

42. Field Notes.—When a survey by azimuths does not include many sights from each set-up, the form given in Fig. 31 is convenient for keeping the notes. The general information concerning the purpose of the survey, the members of the party, etc. is given as for the surveys already described. In the first column of the left-hand page are the courses, and in the next three columns are the lengths, magnetic azimuths, and magnetic bearings, respectively. The two remaining columns of the left-hand page and the entire right-hand page are for remarks and sketches.

Point	Dist.	Magnetic Azimuth	Bearing		
	Transit	at Pt. A			
1	48	130°10′	S 50°E		
8	81	341°0′	N 19°W		
B	659.4	8°44′	N $8\frac{3}{4}$°E		
	Transit	at Pt. B			
1	75	13°40′	N 14°E		
2	112	35°30′	N 35°E		

FIG. 32

The notes read from the bottom of the page upwards. The azimuths of the traverse lines and of important sights are given to the nearest minute; but for locating minor details, as the points on Rattling Run, readings to the nearest 10 minutes are sufficiently close. A complete explanation for the location of point *A* and the method of orienting should be given in the notes as shown. All points should be described fully and accurately. Where possible, the traverse should be closed, as by course *GA*, and, as a check, the azimuth of *AB* should be redetermined by orienting on *G*.

When many points are located from each set-up, the form given in Fig. 32 is commonly used for the left-hand page of the notebook. A sketch, showing the relative locations of the points and what each indicates, should be made on the right-hand page with any other information that cannot be given on the sketch.

SURVEYING BY TRIANGULATION

43. General Principle.—In surveying by triangulation, one line, called a *base line*, is measured, and all other distances are determined by measuring the angles of a triangle and computing the lengths of the sides. The determination of the distance PB in Fig. 22 by measuring the distance PQ and the angles P and Q is an illustration of triangulation. The base line in that case is PQ, the length of which is measured; and, when the angles P and Q are also measured, the distances PB and QB can be calculated by the relations between the sides of a triangle and the sines of the angles. Thus,

$$PB = \frac{PQ \sin Q}{\sin B}$$

and

$$QB = \frac{PQ \sin P}{\sin B}$$

If the distances PB and QB are required with great accuracy, the distance PQ is measured very carefully and the angles are measured by repetition. Moreover, in such a case, the three angles are always measured. A check on the work is thus provided because the sum of the angles should equal 180°.

44. Use of Triangulation.—Triangulation is by far the most accurate and most convenient method of determining distances when the points are very far apart or are separated by rough country. Often, measuring distances in any other way is impracticable because the points may be separated by obstacles which make chaining very difficult; in Fig. 22, the river between P and B is such an obstacle. In surveying by triangulation, it is important that the base line should be as long as possible, and that angles less than 20° should be avoided.

The main use of triangulation, however, is in running a network of lines over a large area in order to locate accurately many points from which other surveys can be started. The most extensive triangulation surveys are being continually made by the United States Coast and Geodetic Survey. In such a survey, the directions of lines are usually referred to the true meridian.

45. Extended Triangulation Survey.—It is not always possible to determine a required distance by means of a single triangle. Then a series of triangles is laid out, in which the base line of each triangle is one of the sides of the preceding triangle. The simplest method of arranging these triangles is shown in Fig. 33; in this case, the process of determining the length and direction of AH is as follows: First, the line BC in any convenient direction is selected as a base line and its length is carefully measured; if desired, the base line can be run through A. Then a number of points, D, E, F, G, are selected so as to form a net of triangles, BCD, CDE, etc., and the three angles of each triangle are measured. Thus, at C, the angles ACB, BCD, and DCE are determined; and at D, the angles BDC, CDE, and EDF are obtained.

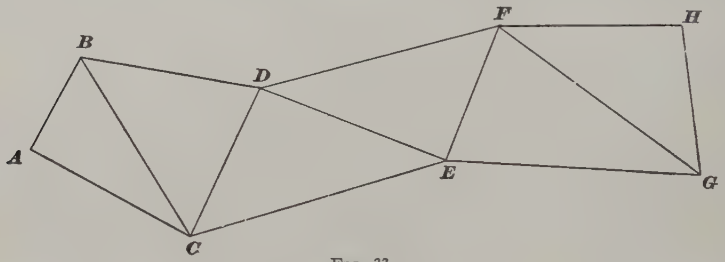


FIG. 33

The azimuth of the base line BC is determined or assumed and then the lengths and azimuths of the other lines can be calculated. The length and azimuth of CD will be found from the measurements in triangle BCD. Then CD can be used as a base line for determining DE from the measurements in triangle CDE. Similarly, DE is the base line for triangle DEF, EF is the base line for triangle EFG, and FG is the base line for triangle FGH. The method of calculating the length and the azimuth of AH will be explained in another Section.

It is evident that all the points should be selected before starting the survey in order that all angles having the same point as a vertex can be measured at the same time. The angles may be taken in any order; for example, it may be more convenient to measure the angles at *C*, *E*, and *G* first, and then to proceed to *B*, *D*, and *F*. It is essential that each point not only should be visible from the other points to which sights are taken but also should be accessible for setting up the transit. In the United States Coast and Geodetic Survey, a tower is often built with a platform at the top when no other suitable point is available.

TRIGONOMETRIC LEVELING

46. Measuring Vertical Angles.—As has been explained, trigonometric leveling is the determination of differences in elevation by means of horizontal or inclined distances and vertical angles. For instance, the difference in elevation between the points *A* and *B*, Fig. 34, may be found by measuring the horizontal or inclined distance from *A* to *B* and the angle between a horizontal line and the line through *A* and *B*. This angle may be measured by means of a transit.

If a transit is in adjustment, the reading of the vertical limb is zero when the line of sight is horizontal. In such a case the vertical angle between the horizontal and the line of sight for any position of the telescope is given directly by means of the reading of the index of the vernier *j′*, Fig. 1, on the vertical limb *i′*.

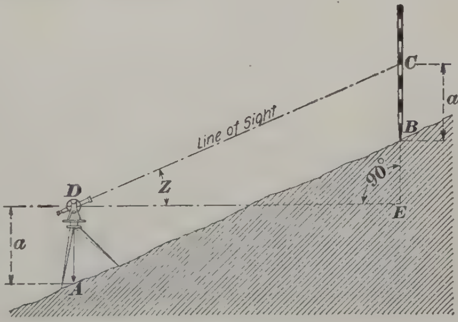


FIG. 34

To measure the vertical angle between the horizontal and a line between two points on the ground, as *A* and *B*, Fig. 34, the transit is set up over one of the points, say *A*, and the vertical distance a from the ground to the center of the transverse axis is measured by means of a tape or a leveling rod. Then a pole or a rod is held vertically at B, and the point C is marked on it at the height a above the ground. Finally, the horizontal cross-wire of the telescope is set on C, and the required vertical angle Z is read from the vertical limb.

If the objective of the telescope is higher than the eyepiece, the angle is an *angle of elevation* and is marked +, or is given without a sign. If the objective is lower than the eyepiece, the angle is an *angle of depression* and is marked −; the sign − must always be written, as a value without any sign is understood to have the sign +.

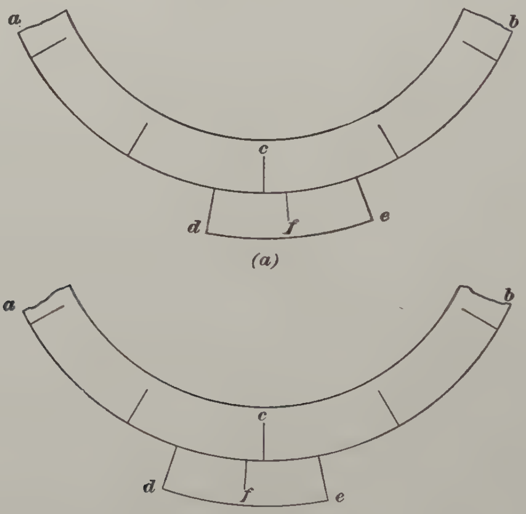


FIG. 35

47. Index Error.—In the method of measuring vertical angles described in the preceding article, it is assumed that the reading of the vertical limb is zero when the line of sight is horizontal. On most transits, the vernier can be shifted slightly on the standard to which it is attached, and the index can thus be set to coincide with the zero of the vertical limb;

but it may not be possible, or perhaps not always convenient, to make this adjustment. In case the vernier is not adjusted, its reading when the bubble of the telescope level is centered is called the index error. Suppose, for instance, that the telescope in Fig. 1 is clamped in a horizontal position, as indicated by the fact that the bubble of the telescope level k' is in the center of its tube, but the index of the vernier j' is not opposite the zero of the vertical limb i'. Let $a\ b$, Fig. 35, represent part of the vertical limb, with the zero point at c; and $d\ e$, the vernier, with the index at f. It is assumed that the telescope is normal and the eyepiece is to the right beyond b.

If the zero of the vernier lies between the zero of the limb and the eyepiece of the telescope in its normal position, as in (a), the index error is marked +; if the zero of the vernier is between the zero of the limb and the objective of the telescope, as in (b), the index error is marked −. When there is an index error, readings of the vertical limb must be corrected by an amount, called the *index correction*, which is equal to the index error. This correction is applied as follows:

Rule I.—*If the error is +, the correction is subtracted from an angle of elevation or added to an angle of depression.*

Rule II.—*If the error is −, the correction is added to an angle of elevation or subtracted from an angle of depression.*

In performing the addition or subtraction, the signs of the angle and of the error are disregarded. For example, if the observed angle is $+10^\circ\ 45'$ and the error is $+4'$, the corrected value of the angle is, by rule I, $10^\circ\ 45' - 4' = 10^\circ\ 41'$. If the angle is $-10^\circ\ 45'$ and the error is $+4'$, the corrected numerical value is $10^\circ\ 45' + 4' = 10^\circ\ 49'$; of course, the negative sign is retained to indicate that the angle is an angle of depression. Again, if the observed angle is $+10^\circ\ 45'$ and the error is $-4'$, the corrected angle is, according to rule II, $10^\circ\ 45' + 4' = 10^\circ\ 49'$. Finally, if the angle is $-10^\circ\ 45'$ and the error is $-4'$, the corrected numerical value is $10^\circ\ 45' - 4' = 10^\circ\ 41'$.

Observations of the index error should be made at intervals during the day, or at least at the beginning and the end of the day's work. It is best to record the actual limb readings and the index error; then the corrected values can be determined in the office at the end of the day.

48. General Formulas.—In Fig. 34, BC is parallel and equal to AD. Hence, $ABCD$ is a parallelogram and DC is parallel and equal to AB. The difference in elevation between A and B is, therefore, represented by the distance CE. From the right triangle CDE,

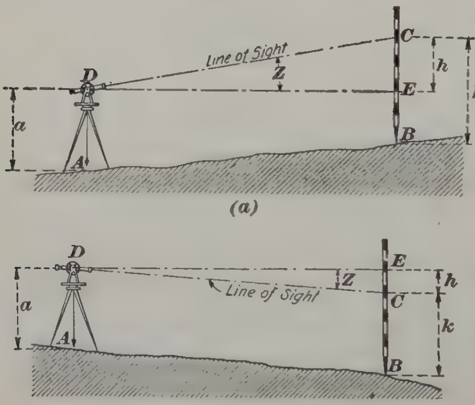
$$CE = DC \sin Z \text{ and } CE = DE \tan Z$$

In general, let

h = difference in elevation between two points;
l = length of inclined line between points;
Z = vertical angle that line makes with horizontal;
s = horizontal distance between points.

Then,
$$h = l \sin Z \qquad (1)$$
$$h = s \tan Z \qquad (2)$$

If the vertical distance is desired with great accuracy, the inclined or the horizontal distance should be measured carefully. Also, the vertical angle should be measured in both directions; for instance, with the transit at A, Fig. 34, a sight is taken to B, and with the transit at B, the reading from B to A is observed. If the two vertical angles differ slightly, the average of the values is taken in calculating the difference in elevation between the points; thus, if the value with the transit at A is +15° 17′, and with the transit at B, it is −15° 19′, the correct value is taken as $\frac{15° 17' + 15° 19'}{2}$, or 15° 18′.



(a)

(b)

FIG. 36

Occasionally, it is more convenient to sight on a point which is not at the same height above the ground as the transverse axis of the instrument; thus, in Fig. 36, BC is not equal to AD.

Let h' = difference in elevation between two points;
a = height of transverse axis above ground at first point;
h = distance computed by formula **1** or formula **2**;
k = rod reading at second point.

If the vertical angle is an angle of elevation, as shown in (*a*), the difference in elevation between A and B is found by the formula

$$h' = a + h - k \qquad (3)$$

If the vertical angle is an angle of depression, as in (*b*), the formula is

$$h' = a - h - k \qquad (4)$$

EXAMPLE 1.—(*a*) If, in Fig. 34, the distance DC is equal to 186.32 feet, the vertical angle from A to B is $+18° 2'$, and the angle from B to A is $-18° 3'$, what is the difference in elevation between A and B? (*b*) If the elevation of A is 110.2 feet, what is the elevation of B?

SOLUTION.—(*a*) The average value of the vertical angle is $\frac{18° 2' + 18° 3'}{2} = 18° 2' 30''$. By formula **1**,

$$h = l \sin Z = 186.32 \times \sin 18° 2' 30'' = 57.7 \text{ ft.} \quad \text{Ans.}$$

(*b*) Since B is higher than A, the elevation of B is equal to $110.2 + 57.7 = 167.9$ ft. Ans.

EXAMPLE 2.—The transit is set up over a point A, Fig. 36 (*b*), with the center of the transverse axis 4.9 feet above the ground; and, with the horizontal cross-wire reading 7.5 feet on a leveling rod held at B, the vertical angle is $-3° 41'$. If the horizontal distance from A to B is 361.74 feet, and the elevation of A is 98.3 feet, find the elevation of B.

SOLUTION.—By formula **2**,

$$h = s \tan Z = 361.74 \tan 3° 41' = 23.3 \text{ ft.}$$

Since the angle Z is negative, C is below E. Then, in formula **4**, $a = 4.9$, $h = 23.3$, and $k = 7.5$; hence,

$$h' = 4.9 - 23.3 - 7.5 = -25.9 \text{ ft.}$$

This means that B is 25.9 ft. below A; the elevation of B is, therefore, equal to $98.3 - 25.9 = 72.4$ ft. Ans.

49. Elevations of Inaccessible Points.—A very useful application of trigonometric leveling is in determining the difference in elevation between two points when one of them is inaccessible. Such a case is shown in Fig. 37, where it is required to find the difference in elevation between A and B, direct leveling and the method explained in Art. 48 being impracticable. If the ground is fairly level for some distance from A, the following procedure is convenient. Set the transit over A, bring the horizontal cross-wire on B, and determine the vertical angle b that the line of sight CB makes with the horizontal line CD; measure also the vertical distance a from the ground to the transverse axis. Next, select a point E on line between A and B, and measure the horizontal distance c. Then, set the transit over E, bring the horizontal wire on B, and determine the vertical angle d between the line of sight FB and the horizontal. Finally, with the transit still at E, find the vertical distance e by setting the line of sight along the horizontal line FG and reading a leveling rod held at A. The difference in elevation between C and F, denoted by f, is equal to $a-e$, and h, the required difference in elevation between A and B, may be computed by the formula

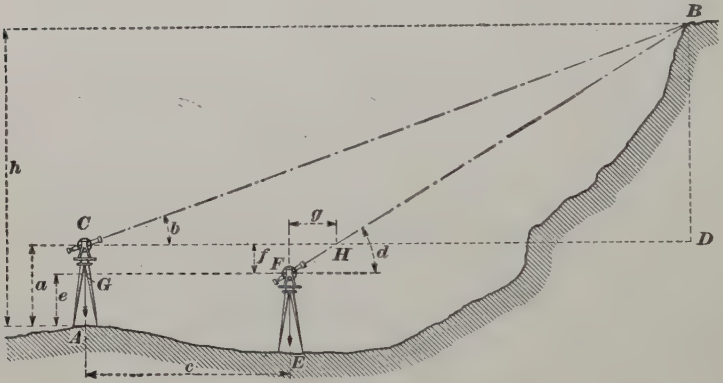


FIG. 37

$$h=a+\frac{c+f\cot d}{\cot b-\cot d} \qquad (1)$$

This relation is derived as follows: The line of sight FB intersects the horizontal line CD at H, the vertical distance between F and H being equal to f, and the horizontal distance g being equal to $f \cot d$. From the figure, $CD=CH+HD$; but $CH=c+g=c+f \cot d$ and $HD=BD \cot d$. Hence,

$$CD=c+f \cot d+BD \cot d$$

Also, in the right triangle BCD,

$$CD=BD \cot b$$

Thus, $$c+f \cot d+BD \cot d=BD \cot b$$

whence, $$BD=\frac{c+f \cot d}{\cot b-\cot d}$$

and $$h=a+BD=a+\frac{c+f \cot d}{\cot b-\cot d}$$

In case the point F is above C, as indicated by the fact that e is greater than a, the value of f is $e-a$ and the formula for the difference in elevation between A and B is

$$h=a+\frac{c-f \cot d}{\cot b-\cot d} \qquad (2)$$

If the horizontal distance CD is required, it may be found by the formula

$$CD=(h-a) \cot b \qquad (3)$$

The point E should be not more than 4 feet vertically below A and not more than 6 feet above A in order that the distance e may be obtained from the set-up at E by reading a rod held on A. In case this is not possible because the ground is too steep, the distance f must be found by direct leveling, a turning point being established between A and E. It is assumed in formulas **1** and **2** that the point B is higher than A; this is almost always the case in practice, because it is hardly possible that B would be visible from both A and E when B is below A.

EXAMPLE.—In order to determine the elevation of a point B, Fig. 37, on the top of a cliff, a transit having an index error of $+3'$ was set over a point A, the elevation of which was known to be 161.8 feet; the height a of the transverse axis was 5.0 feet, and, when the horizontal wire was brought on B, the vertical limb read $+16° 32'$. A point E was next located at a horizontal distance c from A equal to 200 feet. Then the transit was set over E, the horizontal wire brought on B, and the vertical limb reading observed as $+24° 48'$. Finally, with the telescope horizontal, the reading e on a leveling rod held on A was 7.8 feet. Find the elevation of B.

Solution.—Since the index error was +3′, the corrected vertical angle b was 16° 32′−3′, or 16° 29′, and the corrected angle d was 24° 48′−3′, or 24° 45′. The value of f, which is equal to $e-a$, is 7.8−5.0=2.8 ft. Then, the difference in elevation between A and B is found by formula 2. Here,

$$h=5.0+\frac{200-2.8\cot 24°\,45'}{\cot 16°\,29'-\cot 24°\,45'}$$

$$=5.0+\frac{200-2.8\times 2.16917}{3.37955-2.16917}$$

$$=5.0+\frac{193.93}{1.2104}$$

$$=165.2 \text{ ft.}$$

Hence, the elevation of B is 161.8+165.2=327.0 ft. Ans.

50. Often, the method of the preceding article cannot be applied because a suitable point cannot be established between the two given points. In such a case, the following method may be used: Let it be required to determine the difference in elevation between A and B, Fig. 38. First, select a point C several hundred feet from A and in such a position that the point B is visible and the distance AC may be readily chained. Then set the transit over A and, with the vernier reading zero,

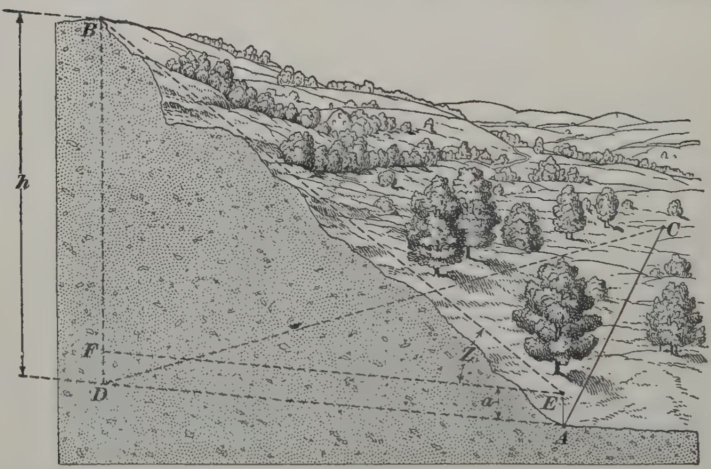


Fig. 38

sight to C. Next, unclamp the upper plate and bring the intersection of the cross-wires on B; for this setting, read the horizontal limb and the vertical limb. Also measure the horizontal distance AC and the vertical distance a from the ground to the transverse axis. Now, set the transit over C and, with the vernier reading zero, sight to A. Finally, loosen the upper clamp, sight to B, and read the horizontal limb. The required difference in elevation is then computed as follows: In the figure, D is a point vertically beneath B and at the same elevation as A. The reading of the horizontal limb when the transit is at A measures the angle between AD and the horizontal projection of AC, and the reading with the transit at C indicates the angle between the horizontal projections of AC and CD. In the triangle ACD, angle ADC is $180° - CAD - ACD$ and

$$AD = \frac{AC \sin ACD}{\sin ADC}$$

The reading of the vertical limb when the transit is at A gives the vertical angle Z between the line of sight EB and the horizontal line EF; from the figure, EF is equal to AD and

$$FB = EF \tan Z = AD \tan Z$$

If Z is an angle of elevation, as in Fig. 38, the difference in elevation between A and B, which is equal to BD and is denoted by h, is $FB + a$ or $AD \tan Z + a$. When the value of AD is substituted, this becomes

$$h = \frac{AC \sin ACD \tan Z}{\sin ADC} + a \qquad (1)$$

If Z is an angle of depression,

$$h = \frac{AC \sin ACD \tan Z}{\sin ADC} - a \qquad (2)$$

If, in formula **2**, h is positive. B is below A; but if h is negative, B is above A.

Example.—The horizontal distance AC, Fig. 33, is 500 feet, angle DAC is 89° 15′, angle ACD is 60° 28′, a is 5.0 feet, and the vertical angle Z is +29° 44′. Find the difference in elevation between A and B.

SOLUTION.—Angle $A\,D\,C$ is equal to $180° - 89°\ 15' - 60°\ 28' = 30°\ 17'$. Then, by formula **1**,

$$h = \frac{A\,C \sin A\,C\,D \tan Z}{\sin A\,D\,C} + a$$

$$= \frac{500 \sin 60°\ 28' \tan 29°\ 44'}{\sin 30°\ 17'} + 5$$

$$= 492.7 + 5 = 497.7 \text{ ft.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. The distance between two points M and N, measured parallel to the ground, is 214.69 feet, and the line of sight parallel to the ground makes a vertical angle of $-11°\ 33'$ from M to N and an angle of $+11°\ 35'$ from N to M. If the elevation of M is 81.6 feet, what is the elevation of N?
Ans. 38.6 ft.

2. The horizontal distance from C to D is 811.6 feet; with the transverse axis 5.2 feet above the ground at C and the horizontal cross-wire set at 4.0 feet on a rod held at D, the vertical angle is $+2°\ 56'$. If the elevation of C is 816.5 feet, what is the elevation of D? Ans. 859.3 ft.

3. In Fig. 37, the angle b is $+7°\ 51'$, the height a is 4.8 feet, the distance c is 250 feet, the angle d is $+12°\ 14'$, and the height e is 3.6 feet. If the elevation of the point A is 37.9 feet, what is the elevation of B?
Ans. 139.5 ft.

4. A base line $A\,B$ has a horizontal length of 400 feet. The transit is set up at A with the transverse axis 4.7 feet above the ground; the horizontal angle from $A\,B$ to an inaccessible point C is 87° 45′, and the vertical angle between a horizontal line and the line of sight from A to C is $-35°\ 24'$. The transit is then set up at B and the horizontal angle $A\,B\,C$ is found to be 65° 30′. If the elevation of A is 1,214.7 feet, what is the elevation of C?
Ans. 644.7 ft.

ADJUSTMENTS OF TRANSIT

51. Conditions of Adjustment.—When a transit is in adjustment, the three following conditions are fulfilled:

1. When the bubbles of the plate levels are in the centers of the tubes, the plates are horizontal, the axis of the instrument is vertical, and the transverse axis of the telescope is horizontal.

2. The line of sight is perpendicular to the transverse axis of the telescope, and, therefore, remains in a vertical plane as the telescope is rotated on the transverse axis.

3. When the bubble of the telescope level is in the center of the tube, the line of sight is horizontal and the vertical limb reads zero.

There are five adjustments that must be made in the same order as they are described in the following articles. An open space which is nearly level and which affords an unobstructed sight of about 450 feet should be chosen; and, in setting up, the tripod legs should be planted firmly in solid ground that is not subject to jars from heavy machinery or other causes.

52. First Adjustment.—First, it is necessary to make the axes of the plate levels perpendicular to the vertical axis of the instrument in order that, when the bubbles are centered by the leveling screws, the axis will be vertical and the plates will revolve in a horizontal plane. This adjustment, which is substantially the same as for the compass, is performed as follows: With the upper clamp set and the lower clamp loose, turn the instrument so that each plate level will be parallel to the line determined by a pair of opposite leveling screws; then bring each bubble to the middle of its tube by means of the corresponding pair of leveling screws. Next, revolve the instrument on its vertical axis through 180°. If the levels are in adjustment, the bubbles will remain in the centers of the tubes. If either bubble runs toward one end of the tube, bring it half-way back to the center of the tube by means of the capstan-headed screw at one end of the tube; then bring the bubbles to the centers by means of the leveling screws. Repeat the operation until both bubbles remain in the centers of the tubes in both positions of the instrument.

53. Second Adjustment.—The next operation is to make the line of sight perpendicular to the transverse axis of the telescope. Set up the transit near the center of the open space, as at *A* in Fig. 39. Then with the telescope normal, sight on some well-defined point *B*, a few hundred feet distant, using the point of intersection of the cross-wires. Both plates being clamped, plunge the telescope and set another point a few hundred feet from the instrument; a mark on a wall or fence is very convenient, or a wide stake can be driven on line and the point marked on it. If the line of sight is perpendicular to the transverse axis, this point will be at *C* in the prolongation of *BA*.

In order to ascertain whether this is the case, unclamp either plate, rotate the instrument in azimuth with the telescope still reversed, and sight to *B* again; then plunge the telescope back to normal. If the line of sight strikes the same point as before, no adjustment is necessary. If the line of sight does not pass through the first point, mark a second point on the same object so that the two points will be very nearly the same distance from the instrument. Suppose *D* is the point set after the first plunging and *E* is the second point. Then on the line between *D* and *E*, mark point *F*, making the distance *EF* from the second point *E* equal to one-quarter of *DE*. Move the cross-wires by means of the capstan-headed screws on the sides of the telescope, as described for the wye level, until the line of sight passes through *F*. For an erecting telescope, loosen the screw toward which the image of the wire should be moved and tighten the opposite screw; for an inverting telescope, loosen the screw away from which the image is to be moved and tighten the other. Thus, for the assumed conditions, loosen the left-hand screw for the erecting telescope or the right-hand screw for the inverting telescope. Repeat the operations until the line of sight after the second plunging passes through the point marked after the first sight.

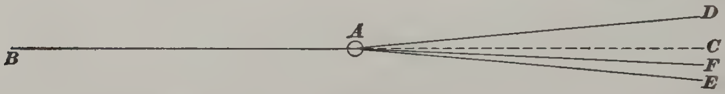


FIG. 39

54. Third Adjustment.—Now the transverse axis of the telescope is made perpendicular to the vertical axis of the instrument in order that, when the instrument is leveled, the transverse axis will be horizontal. This adjustment is made best by sighting, with the telescope normal, to some well-defined point on a high object such as a church spire. In this case, also, the point of intersection of the cross-wires must be used for all sights. Having clamped both plates, depress the objective and set a point on the ground in the line of sight and as far as possible from the instrument. Unclamp one plate, revolve the instrument on its vertical axis, and, with the telescope reversed, sight again to the high point. If, when the telescope is depressed, the line of sight passes through the first point on the ground, no adjustment is required. Otherwise, set a second point near the first one. Suppose that *A*, Fig. 40, is the high point, *B* is the first point on the ground, and *C* is the second point on the ground. Then mark point *D* half-way between *B* and *C* and bring the line of sight to pass through the point *D* by rotating the telescope on the vertical axis. Next, point the telescope upwards; the line of sight will not pass through *A*, but will strike to one side, say at *E*. Finally, bring the vertical wire on *A* by adjusting the capstan-headed screw underneath one end of the transverse axis. If the line of sight is to move to the right, as from *E* to *A* in Fig. 40, lower the right end of the axis or raise the left end, keeping the telescope in its reversed position; if the line of sight is to move to the left, raise the right end or lower the left end. Repeat the operation until the line of sight in the second position of the telescope passes through the point on the ground determined by the first position. Since the vertical cross-wire is in adjustment, the transverse axis is adjusted in the same manner for both an erecting and an inverting telescope. In case the position of the transverse axis is altered, it is necessary to test again the adjustment of the vertical cross-wire.

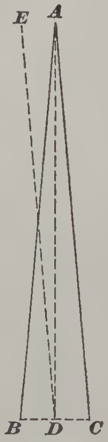


FIG. 40

55. Supplementary Test.—In order to make the cross-wires vertical and horizontal, and thus make it unnecessary to bring the point of intersection of the cross-wires on the point sighted at, the following test is convenient: Level the instrument carefully. Then bring one end of the vertical wire on some well-defined point, and revolve the telescope on its transverse axis so that the point appears to move along the wire. If the point does not remain on the wire throughout the motion, loosen two adjacent screws and rotate the ring holding the wires. Then, repeat the operations. This test should be performed after the transverse axis has been adjusted.

56. Fourth Adjustment.—The line of sight should be horizontal when the bubble of the telescope level is centered, in order that the transit may be accurate for leveling and for the measurement of vertical angles. The adjustment consists of two parts; first the horizontal cross-wire is adjusted, and then the telescope level is tested.

To adjust the horizontal cross-wire, set up at a point *A*, Fig. 41, and drive pegs at points *B* and *C* in a straight line from *A*, and respectively about 15 and 200 feet from *A* in the same direction. Clamp the transverse axis of the telescope so that the line of sight is about parallel to the ground, and read a leveling rod held on the pegs at *B* and *C*. Then plunge the telescope, revolve the instrument on its vertical axis, and clamp the transverse axis so that the horizontal cross-wire cuts the same reading on the rod at *B* as it did before. If the rod reading on the peg at *C* agrees with the reading previously obtained, no adjustment is necessary. If the two readings do not agree, bring the horizontal cross-wire to read the average of the two values by means of the capstan-headed screws on top and bottom of the telescope, keeping the telescope itself clamped in position. The method of moving the cross-wire is the same as described for a level.

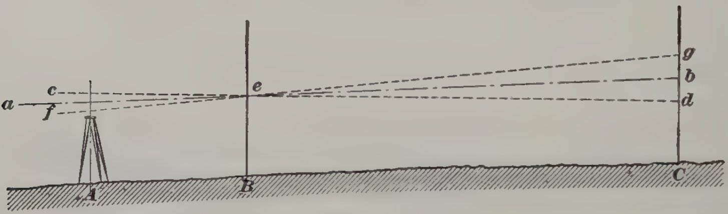


FIG. 41

For the sake of clearness, the conditions are shown greatly exaggerated in Fig. 41, and the telescope of the transit is omitted. Suppose that $a\,b$ represents the theoretical line of sight for an instrument in adjustment; $c\,d$, the actual line of sight for the first position of the telescope, e being the point cut on the rod at B; and $f\,g$, the line of sight for the second position of the telescope, this line being made to pass through e. Then the point b, midway between d and g, is marked, and the horizontal wire is set on that point by means of the capstan-headed screws..

The adjustment is tested by taking new readings on the rods at B and C for the two positions of the telescope. If the readings at C do not agree yet, the wire must be moved again. The operations are repeated as often as necessary.

57. The telescope level is adjusted by the peg method described for the dumpy level. All rod readings are taken with the bubble of the telescope level in the center of its tube. Suppose the rod readings on the pegs from the first set-up are r_1 and r_2, and the rod readings from the second set-up are r_3 and r_4. Then if $r_3-r_4=r_1-r_2$, the level is in adjustment. If r_3-r_4 is not equal to r_1-r_2, the corrected value of r_4 is determined as explained in *Leveling*. The telescope is then rotated on its transverse axis so that the horizontal cross-wire reads the corrected value of r_4; the telescope bubble will, therefore, move from the center of its tube. With the telescope clamped in that position, the bubble is brought to the center of the tube by means of the capstan-pattern nuts l', Fig. 1. The operation is repeated until $r_3-r_4=r_1-r_2$.

58. Fifth Adjustment.—The final adjustment is to make the vernier of the vertical limb read zero when the line of sight is horizontal. Set up the instrument and bring the bubble of the telescope level to the center of the tube. Then, if the vernier does not read zero, set it to zero by shifting it on the standards by whatever means are provided for the purpose.

OFFICE WORK IN ANGULAR SURVEYING

Serial 3070-2 Edition 1

LATITUDES AND DEPARTURES

PRELIMINARY EXPLANATIONS

1. Introduction.—Office work in angular surveying consists of the solutions of three main problems: (1) balancing a survey, or correcting the field measurements to distribute unavoidable errors among the courses; (2) plotting the courses of a survey; and (3) computing the area of the land bounded by a closed traverse.

2. Reference Axes.—For the purposes of calculating and plotting, it is convenient to refer the directions of survey courses and the locations of survey points to two lines at right angles to each other; these lines are called reference axes. In Fig. 1, TT' and GG' are reference axes intersecting at the point O. Usually, the axis TT' represents either the true or the magnetic meridian through some point of the survey, and the axis GG' is an east-and-west line. If TT' is not a true or magnetic meridian, it is called an assumed meridian.

3. Latitudes and Departures of Courses.—Ordinarily the relative locations of two points are determined by the length and direction of the straight line between the points. For instance, in Fig. 1, the point Q may be located with respect to P by the length and direction of the line PQ. But the point Q may also be located from the point P by laying off from

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P a given distance PD parallel to TT' and from D a given distance DQ parallel to GG'. Thus, one point may be located from another by constructing a right triangle, the legs of which are parallel to a pair of mutually perpendicular reference axes.

In the case of the courses of a survey, the legs of all the right triangles are drawn parallel to the same pair of reference axes. For example, in Fig. 1, let TT' and GG' represent reference axes at right angles to each other, and PQ and $P'Q'$ two courses of a survey. Then, the lines PD, QE, $P'D'$, and $Q'E'$ are drawn parallel to the axis TT', and the lines QD, PE, $Q'D'$, and $P'E'$ are made parallel to GG'. In the triangle constructed for a given course, the length of the leg parallel to the meridian is called the *latitude* of the course and the length of the leg at right angles to the meridian is the *departure* of the course. In Fig. 1, PD or EQ is the latitude of PQ, and PE or DQ is the departure of PQ. Similarly, $P'D'$ or $E'Q'$ is the latitude of $P'Q'$, and $P'E'$ or $D'Q'$ is the departure of $P'Q'$.

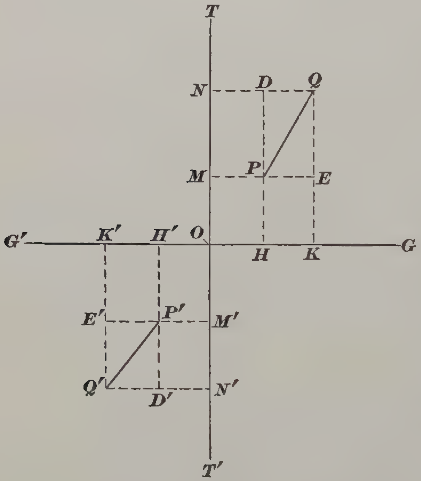


FIG. 1

The latitude and the departure of PQ are also represented by the distances MN and HK. Likewise, $M'N'$ and $H'K'$ are the latitude and the departure of $P'Q'$. In other words, the latitude of a course is the distance measured along the meridian, between lines drawn through the ends of the course at right angles to the meridian; the departure of a course is the distance measured along a line perpendicular to the meridian, between lines drawn through the ends of the course parallel to the meridian.

Latitude is sometimes called *latitude range* or *latitude difference*, and departure is sometimes referred to as *longitude range* or *longitude difference.*

4. Signs of Latitude and Departure.—In order to indicate the direction of a course with respect to a pair of reference axes, positive and negative quantities are used. If the bearing of a course with respect to the reference meridian is northeast or northwest, the latitude of the course is considered positive and is called a *northing*. If the bearing of a course is southeast or southwest, the latitude is negative and is known as a *southing*. For instance, in Fig. 1, the latitude of PQ is $+MN$ or simply MN, and the latitude of $P'Q'$ is $-M'N'$.

If the bearing of a course is northeast or southeast, the departure of the course is assumed to be positive, and is designated as an *easting*. If the bearing of a course is northwest or southwest, its departure is negative and is called a *westing*. Thus, in Fig. 1, the departure of PQ is HK and the departure of $P'Q'$ is $-H'K'$.

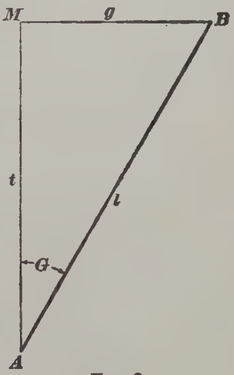


FIG. 2

COMPUTATIONS INVOLVING LATITUDES AND DEPARTURES

5. General Formulas.—In Fig. 2, let AM represent the direction of a reference meridian; MB, a line at right angles to the meridian; AB, a given course; and G, the angle that the course makes with the meridian. Then, from trigonometry,

$$AM = AB \cos G, \text{ and } MB = AB \sin G$$

But AM may be considered as the latitude of the course, and MB, its departure. Thus, the latitude and the departure of a course may be found by the following rules:

Rule I.—*The latitude of a course is equal to the product of the length of the course and the cosine of the angle between the meridian and the course.*

Rule II.—*The departure of a course is found by multiplying the length of the course by the sine of the angle that the course makes with the meridian.*

In general, let

l = length of course;
G = angle between meridian and course:
t = latitude of course;
g = departure of course.

Then,

$$t = l \cos G \qquad (1)$$
$$g = l \sin G \qquad (2)$$

When the latitude and the departure of a course are given, the angle between the meridian and the course, and the length of the course can be computed by the following formulas:

$$\tan G = \frac{g}{t} \qquad (3)$$

$$l = \frac{g}{\sin G} \qquad (4)$$

$$l = \frac{t}{\cos G} \qquad (5)$$

$$l = \sqrt{t^2 + g^2} \qquad (6)$$

These relations hold good for any direction of the course and also apply whether the angle G is given by a bearing or by an azimuth.

6. Given Length and Bearing.—In the case of bearings, the angle between the meridian and a course is always given as less than 90°. Hence, the functions of G are readily found. The numerical values of t and g may then be calculated by formulas **1** and **2**, Art. **5**. They should contain the same number of decimal places as the given value of l. The signs of t and g are determined from the quadrant of the bearing by the following rules:

Rule I.—*If the bearing is northeast, both the latitude and the departure are positive*

Rule II.—*If the bearing is southeast, the latitude is negative and the departure is positive.*

Rule III.—*If the bearing is southwest, both the latitude and the departure are negative.*

Rule IV.—*If the bearing is northwest, the latitude is positive and the departure is negative.*

It is a common mistake to reverse the latitude and the departure. This will be avoided, or easily detected, if it is kept in mind that for angles less than 45°, the cosine is greater than the sine, and for angles greater than 45°, the sine is greater than the cosine. Therefore, for bearings less than 45°, the latitude of a course is greater than its departure; and for bearings greater than 45°, the departure is greater than the latitude. It is obvious that the latitude of a north-and-south line is equal to the length of the line, and its departure is zero; likewise, the departure of an east-and-west line is equal to the line itself, and the latitude is zero. It will be noticed that the latitude and the departure depend only on the direction of the meridian. They are, therefore, the same for any pair of axes that are parallel and perpendicular to the meridian.

EXAMPLE.—The length of a course is 896.7 feet and its bearing is N 39° 15′ W; find the latitude and the departure.

SOLUTION.—Here $l=896.7$ and $G=39° 15'$. Then, by formulas **1** and **2**, Art. **5**, the numerical values of t and g are

$$t=896.7 \cos 39° 15'$$
$$g=896.7 \sin 39° 15'$$

If natural functions are used,

$$t=896.7\times .77439=694.4 \text{ ft.}$$
$$g=896.7\times .63271=567.4 \text{ ft.}$$

Since the bearing of the course is northwest, rule IV is used for determining the signs of t and g; thus, t is positive and g is negative. The latitude is, therefore, 694.4 ft. and the departure is -567.4 ft. Ans.

Unless the numbers are comparatively easy to multiply, it is preferable to use logarithms. The logarithmic work is conveniently arranged by writing the logarithm of the length with the logarithmic sine of the bearing above it and the logarithmic cosine of the bearing below it; then the addition is performed upwards in one case and downwards in the other. In this problem the work will appear as follows:

$$\log g = 2.75385 \qquad g = 567.4 \text{ ft.}$$

$$\begin{aligned} \log \sin 39^\circ\ 15' &= 9.80120 \\ \log 896.7 &= 2.95265 \\ \log \cos 39^\circ\ 15' &= 9.88896 \end{aligned}$$

$$\log t = 2.84161 \qquad t = 694.4 \text{ ft.}$$

The signs of t and g are found by applying rule IV. Thus, $t=694.4$ ft. and $g=-567.4$ ft. Ans.

7. Given Length and Azimuth.—When the angle between the meridian and a course is given by the azimuth of the course, the angle may be greater than 90°. In this case, the latitude and the departure may be determined in two ways.

In one method, the azimuth is changed to a corresponding bearing, and the latitude and the departure are then computed from the length and the bearing of the course. The various conditions are shown in Fig. 3. Let NS represent the meridian; AB, a given course; Z, its azimuth; Z', its bearing; l, its length; t, its latitude; and g, its departure. In (a), the azimuth is less than 90°; thus, the bearing is northeast and the angle is also equal to Z. In (b), the azimuth is between 90° and 180°; for this case, the bearing is southeast and the angle Z' is $180°-Z$. In (c), the azimuth is between 180° and 270°; the bearing is southwest and the angle Z' is $Z-180°$. In (d), the azimuth is between 270° and 360°; here, the bearing is northwest and the angle Z' is $360°-Z$. In each case the latitude

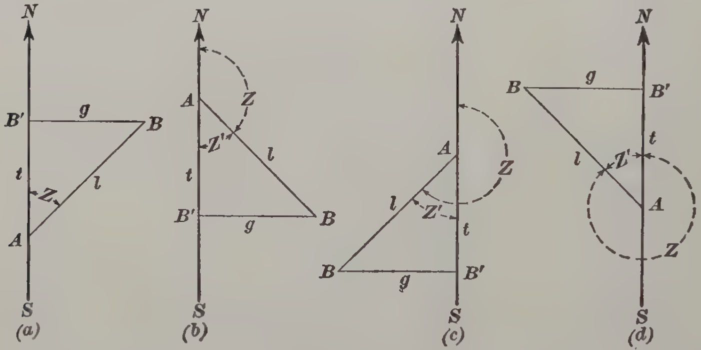


FIG. 3

and the departure with the proper signs may be determined as explained in the preceding article.

In the second method, the latitude and the departure are calculated directly from the length and the azimuth by applying the principles of trigonometry. In Fig. 4, the plane around the point O is divided into quadrants by the meridian NS and the east-and-west line EW. When azimuths are considered, the northeast quadrant NOE is the first quadrant; the southeast quadrant EOS, the second; the southwest quadrant SOW, the third; and the northwest quadrant WON, the fourth. Thus, the course OA, the azimuth Z_1 of which is less than 90°, is in the first quadrant; the course OB, having an azimuth Z_2 between 90° and 180°, is in the second quadrant; the azimuth Z_3 of OC is between 180° and 270°, and the course is in the third quadrant; and, since the azimuth Z_4 of OD is between 270° and 360°, the course is in the fourth quadrant. Although this method of numbering the quadrants differs from that adopted in trigonometry with regard both to the starting line and to the direction in which the numbers increase, the functions of angles have the same signs in both systems because distances to the right and distances upwards are considered as positive in each case. The values of t and g for the conditions shown in Fig. 3 will now be determined by trigonometry.

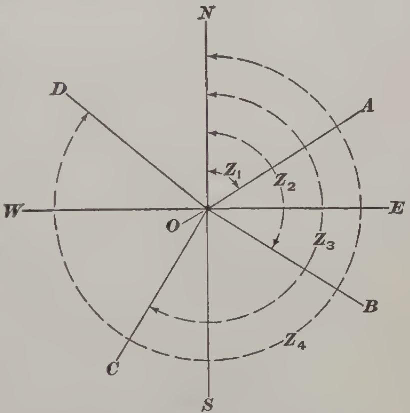


FIG. 4

In (a), the course AB is in the first quadrant and the angle Z is acute; sin Z and cos Z are both positive, and, therefore, the latitude and the departure of AB are also positive. In (b), AB is in the second quadrant, and the acute angle from which the functions of Z are derived is equal to $Z' = 180° - Z$. Then, $\sin Z = \sin (180° - Z') = \sin Z'$, and $\cos Z = \cos (180° - Z')$

$= -\cos Z'$; consequently, the departure is positive and the latitude is negative. In (*c*), AB is in the third quadrant and $Z' = Z - 180°$. In this case, $\sin Z = \sin(180° + Z') = -\sin Z'$, and $\cos Z = \cos(180° + Z') = -\cos Z'$; the latitude and the departure are both negative. In (*d*), AB is in the fourth quadrant and $Z' = 360° - Z$. Here, $\sin Z = \sin(360° - Z') = -\sin Z'$, and $\cos Z = \cos(360° - Z') = \cos Z'$; therefore, the latitude is positive and the departure is negative.

The calculations in both methods are practically the same but the reasoning is slightly different; it is unimportant which is used. As in the case of bearings, the number of decimal places in the values of t and g should be taken the same as in the given length.

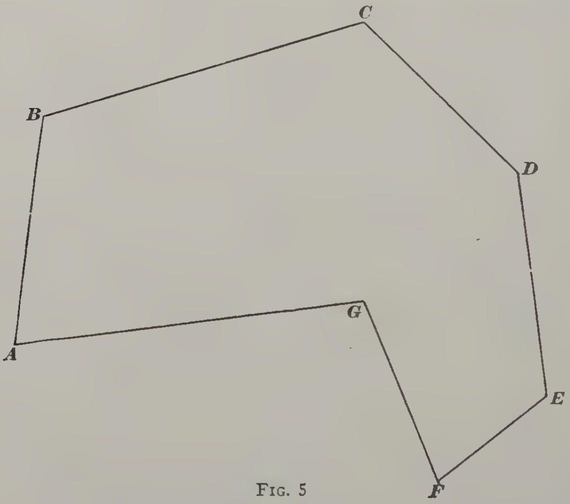


FIG. 5

EXAMPLE.—Find the latitude and the departure of a course having a length of 431.45 feet and an azimuth of 231° 9′.

SOLUTION BY BEARINGS.—Since the azimuth is between 180° and 270°, the bearing is southwest. Hence, the latitude and the departure are both negative. The angle G is equal to $231°\ 9' - 180° = 51°\ 9'$. Then, by formulas **1** and **2**, Art. **5**,

$$t = l \cos G = 431.45 \cos 51°\ 9' = 270.64 \text{ ft}$$
$$g = l \sin G = 431.45 \sin 51°\ 9' = 336.01 \text{ ft.}$$

Therefore, the latitude is -270.64 ft. and the departure is -336.01 ft. Ans.

It will be noticed that the departure is greater than the latitude because G is greater than 45°.

SOLUTION BY TRIGONOMETRY.—The course lies in the third quadrant and the acute angle from which the functions of the given angle are derived is equal to $231° 9' - 180° = 51° 9'$. Thus, $\sin 231° 9' = \sin (180° + 51° 9') = -\sin 51° 9'$ and $\cos 231° 9' = \cos (180° + 51° 9') = -\cos 51° 9'$. Then, by formulas **1** and **2**, Art. **5**,

$$t = l \cos 231° 9' = -431.45 \cos 51° 9' = -270.64 \text{ ft.} \quad \text{Ans.}$$
$$g = l \sin 231° 9' = -431.45 \sin 51° 9' = -336.01 \text{ ft.} \quad \text{Ans.}$$

8. Values for Survey Courses.—The latitudes and the departures of the courses of the transit survey shown in Fig. 5, with other necessary data, are recorded in Table I. In the first three columns are given the courses, the azimuths, and the lengths, which are copied from the field notes. In the fourth and fifth columns are the latitudes of the courses, and in the last two columns are the departures. In order to avoid confusion of signs and also for convenience in balancing the survey, the northings and the southings are placed in separate columns; similarly, the eastings and the westings are separated.

TABLE I

LATITUDES AND DEPARTURES

Course	Azimuth	Length	Latitude		Departure	
			N	S	E	W
AB	8° 51′	659.43	651.59		101.45	
BC	70° 42′	897.46	296.62		847.02	
CD	132° 45′	594.01		403.22	436.20	
DE	170° 31′	628.37		619.79	103.53	
EF	228° 36′	387.52		256.27		290.68
FG	335° 55′	547.55	499.89			223.44
GA	260° 13′	988.99		168.05		974.60

9. Given Latitude and Departure, To Find Length and Direction.—Frequently, the latitude and the departure of a course are known and it is required to determine the length and the bearing or azimuth of the course. If the values of t and g are easy to square or if the direction is not needed, the length may be found conveniently by formula **6**, Art. **5**, which is $l=\sqrt{t^2+g^2}$. In other cases, the angle G is found first by formula **3**, which is $\tan G=\frac{g}{t}$; then l is determined from either formula **4**, which is $l=\frac{g}{\sin G}$, or formula **5**, which is $l=\frac{t}{\cos G}$.

When formula **3** is used, the value of the acute angle G is first computed by temporarily disregarding the signs of t and g, that is, by considering that both are positive. Then the quadrant in which the course lies is determined by considering the given signs of t and g and applying the following rules:

Rule I.—*If t and g are both positive, the bearing is northeast, and the azimuth is equal to G.*

Rule II.—*If g is positive and t is negative, the bearing is southeast, and the azimuth is equal to 180°−G.*

Rule III.—*If t and g are both negative, the bearing is southwest, and the azimuth is equal to 180°+G.*

Rule IV.—*If t is positive and g is negative, the bearing is northwest, and the azimuth is equal to 360°−G.*

In formulas **4**, **5**, and **6**, the signs of g and t are unimportant since the length of the course is always considered positive. When an angle is small, a little difference in the value of the angle has practically no effect on the cosine but changes the sine considerably. Therefore, for values of G less than 20°, formula **5** is more accurate than formula **4**. On the other hand, for angles near 90°, a small difference in the value of the angle affects the cosine much more than the sine. Consequently, when G is greater than 70°, formula **4** is to be used rather than formula **5**. For values of G between 20° and 70°, formula **4** or formula **5** may be employed.

When the given latitude and departure are measured to hundredths of a foot, it is sufficiently accurate to take G

to the nearest minute. If the given distances are to thousandths of a foot, G should be taken to the nearest 10 seconds. In any case, the number of decimal places in the calculated length should be taken the same as in the given latitude and departure.

EXAMPLE 1.—The latitude and the departure of a course are, respectively, -13.71 chains and 9.38 chains. What are the bearing and the length of the course?

SOLUTION.—To find G, disregard the signs of g and t and use formula **3**, Art. **5**. In this case, g is 9.38 and t is taken as 13.71, the negative sign being neglected temporarily. Then,

$$\tan G = \frac{g}{t} = \frac{9.38}{13.71}, \text{ and } G = 34° \; 23'$$

By rule II, the bearing is southeast, because t is negative and g is positive. Hence, the bearing is S 34° 23′ E. Ans.

Finally, by formula **4**, Art. **5**,

$$l = \frac{g}{\sin G} = \frac{9.38}{\sin 34° \; 23'} = 16.61 \text{ ch.} \quad \text{Ans.}$$

The logarithmic work is as follows:

$$\begin{array}{rl} \log \; 9.38 &= 0.97220 \\ \log \; 13.71 &= \underline{1.13704} \\ \log \tan G &= 9.83516 \\ G &= 34° \; 23' \end{array} \qquad \begin{array}{rl} \log 9.38 &= 0.97220 \\ \log \sin 34° \; 23' &= \underline{9.75184} \\ \log l &= 1.22036 \\ l &= 16.61 \text{ ch.} \end{array}$$

First, the difference between the logarithm of 9.38 and the logarithm of 13.71 is the logarithmic tangent of G, from which G is determined. At the same time that G is taken out of the table, its logarithmic sine is obtained. It is subtracted from the logarithm of 9.38 to get the logarithm of l. For the purpose of finding l, G is taken to the nearest minute; however, in a compass survey, where bearings are estimated to the nearest 5 minutes, the bearing of the line would be given as S 34° 25′ E.

EXAMPLE 2.—The latitude of a course is $+125.04$ feet and the departure is -216.81 feet; find the azimuth and the length.

SOLUTION.—By formula **3**, Art. **5**,

$$\tan G = \frac{216.81}{125.04}, \text{ and } G = 60° \; 2'$$

Since t is positive and g is negative, rule IV applies; hence, the azimuth is $360° - 60° \; 2' = 299° \; 58'$. Ans.

By formula **4**, Art. **5**,

$$l = \frac{g}{\sin G} = \frac{216.81}{\sin 60° \; 2'} = 250.26 \text{ ft.} \quad \text{Ans.}$$

The logarithmic work is as follows:

$\log 216.81 = 2.33608$
$\log 125.04 = 2.09705$
$\log \tan G = 0.23903$
$G = 60° \ 2'$

$\log 216.81 = 2.33608$
$\log \sin 60° \ 2' = 9.93768$
$\log l = 2.39840$
$l = 250.26$ ft.

10. Total Departures and Total Latitudes.—In the foregoing articles, the lengths of the courses of a survey and their directions with respect to reference axes are given by means of departures and latitudes. Reference axes are also used for the purpose of locating the points of a survey. In Fig. 6, the point A may be located by the distance CA perpendicular to the meridian TT' and the distance DA perpendicular to the axis GG'. The perpendicular distance from the meridian to a point is called the *total departure* of the point, and the perpendicular distance from the east-and-west axis to the point is its *total latitude*. Thus, the total departure of B is the distance EB perpendicular to TT', and the total latitude of B is the distance FB perpendicular to GG'.

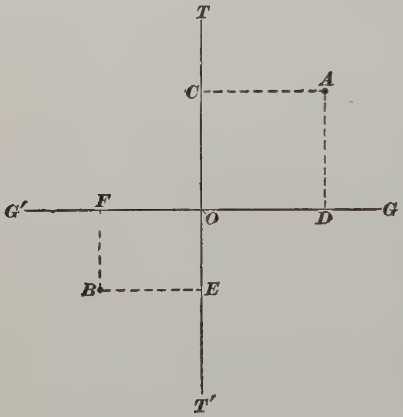


FIG. 6

The total latitude and the total departure of A are also equal to the distances OC and OD, respectively; and the values for B are OE and OF. That is, the total latitude of a point is the distance measured along the meridian from the intersection of the reference axes to the foot of the perpendicular from the point to the meridian. The total departure of a point is the distance measured along the east-and-west axis from the intersection of the axes to the foot of the perpendicular from the point to the axis.

The departure and latitude of a course are distances parallel to the reference axes; and the total departure and total latitude of a point are also distances parallel to the axes. It must be kept in mind, therefore, that the terms departure and latitude refer to a line, whereas total departure and total latitude refer to a point.

The total departure is said to be *east* or *west* according as the point is east or west of the meridian; the total latitude is *north* or *south* according as the point is north or south of the east-and-west axis. North total latitudes and east total departures are considered *positive*, and south total latitudes and west total departures are *negative*. In Fig. 6, the total latitude of A is OC or DA, and that of B is $-OE$ or $-FB$; the total departure of A is OD or CA, and the total departure of B is $-OF$ or $-EB$.

11. Determination of Total Latitudes and Total Departures.—For computing the total latitudes and the total departures of the points of a survey, the reference axes are chosen through some corner of the survey; both values for this point are, therefore, equal to zero. For example, in Fig. 7, the axes GG' and TT' pass through the corner A; the total latitude and the total departure of A are then equal to zero. The total latitude of the point B is EB, which is equal to the latitude of the course AB. The total latitude of the point C is HC, which is equal to $HK+KC$; but since HK is equal to EB or the total latitude of the point B, and KC is the latitude of the course BC, the total latitude of C is equal to the total latitude of B plus the latitude of the course BC. The total latitude of D is $-LD$, which may be considered as equal to $HC-CM$; thus, the total latitude of D is equal to the total latitude of C minus the latitude of CD. The total departure of

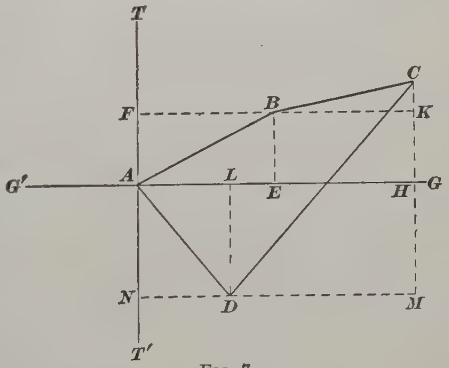


FIG. 7

a point can be found in a similar manner. The computations may, therefore, be made by applying the following rules:

Rule I.—*The total latitude of a point is equal to the total latitude of the preceding point plus or minus the latitude of the course between them. If the latitude of the course is a northing, it is added; and if it is a southing, its numerical value is subtracted.*

Rule II.—*The total departure of a point is equal to the total departure of another point plus or minus the departure of the course between them. If the departure of the course is an easting, it is added; and if it is a westing, it is subtracted.*

The following rules for addition and subtraction will be helpful:

Rule III.—*If both numbers have the same sign, add the numerical values and prefix the common sign.*

Rule IV.—*If the numbers have different signs, subtract the smaller numerical value from the larger and prefix the sign of the larger.*

For example, according to rule III, $+3+2=+5$ and $-4-6=-10$. By rule IV, $+5-3=+2$, $-3+6=+3$, and $+5-6=-1$.

Computations for determining total latitudes and total departures are shown in the following example. In all cases, it is advisable to make a diagram of the conditions.

Example.—The latitudes and the departures of the courses in Fig. 8 are given in the following table. Find the total latitudes and the total departures of B, C, D, and E with respect to axes GG' and TT' through A.

Course	Latitude	Departure
AB	+216	+153
BC	− 97	+271
CD	−244	− 59
DE	−100	−500

Solution.—The total latitude and the total departure of A are evidently equal to zero.

The total departure of *B* is, by rule II, equal to the total departure of *A* plus the departure of *A B*, or 0+153=153. Ans.

The total departure of *C* is equal to the total departure of *B* plus the departure of *B C*, or 153+271=424. Ans.

The total departure of *D* is equal to the total departure of *C* minus the departure of *C D*, or 424−59=365. Ans.

The total departure of *E* is equal to the total departure of *D* minus the departure of *D E*, or 365−500. The difference between the numerical values is 500−365=135; since the larger number has a negative sign, the result is −135. Ans.

This negative sign indicates that *E* is west of the meridian through *A*. The total latitudes are found in a similar manner by applying rule I.

For *B*, 0+216=216. Ans.
For *C*, 216−97=119. Ans.
For *D*, 119−244=−125. Ans.
For *E*, −125−100=−225. Ans.

The negative signs for *D* and *E* indicate that these points are south of *A*.

12. In order to compute the total latitude and the total departure of *E*, Fig. 8, by the preceding method, it was necessary to determine the values for the points *B*, *C*, and *D* also. Often, the locations of these intermediate points are not needed. In such a case, it is more convenient to apply the following rules:

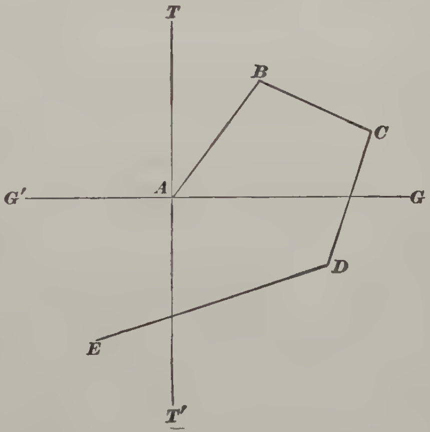


FIG. 8

Rule I.—*To find the total latitude of any point, take the sum of the northings and the sum of the southings of the courses between the starting point and the point in question; find the difference between these sums. If the northings exceed the southings, add this difference to the total latitude of the starting point; if the southings are greater than the northings, subtract the difference from the given total latitude.*

Rule II.—*To find the total departure of a point, take the difference between the sum of the eastings and the sum of the westings. Add the difference to, or subtract it from, the total departure of the starting point according as the eastings or the westings are greater.*

EXAMPLE.—Using the values given in the example in Art. **11**, find the total latitude and the total departure of E with respect to axes through A.

SOLUTION.—The only northing is 216, and the sum of the southings is $97+244+100=441$. The difference between these sums is $441-216=225$. Since the southings are greater than the northings, the difference is subtracted from the total latitude of A, which is zero. Hence, the total latitude of E is $0-225=-225$. Ans.

The sum of the eastings is $153+271=424$, and the sum of the westings is $59+500=559$. The difference between the sums, which is $559-424=135$, is subtracted because the westings are greater. Hence, the total departure of E is $0-135=-135$. Ans.

13. Latitude and Departure of Line From Total Latitudes and Total Departures of Its Ends.

—In many cases, the total latitudes and the total departures of two points are known, and it is required to find the latitude and the departure of the course between the points. For these calculations, the following rule may be applied:

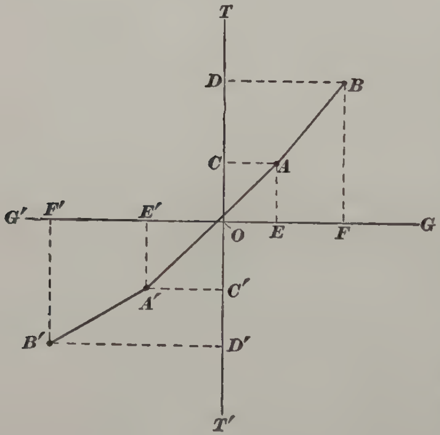


FIG. 9

Rule.—*The latitude, or the departure, of a course is equal to the difference between the total latitudes, or the difference between the total departures of the extremities of the course.*

For example, in Fig. 9, the latitude of the course AB is equal to the total latitude of B minus the total latitude of A; that is, $CD=OD-OC$. Similarly, the latitude of the course $A'B'$ is equal to the total latitude of B' minus the total latitude of A', or $C'D' = OD' - OC'$. To find the latitude of a course one end of which is north of the reference axis and the other end of which is south of the axis, the negative sign of one of the values must be considered. Thus, to determine the latitude of AA' by the preceding rule, the total latitude of A is taken as $+OC$ and that of A' as $-OC'$. Their difference may then be indicated as $OC - (-OC')$. The rule for subtracting a negative number from a positive one is to change the sign of the subtrahend and add it to the minuend. Hence, $OC - (-OC') = OC + OC'$. Since $OC + OC'$ is equal to CC', which is the latitude of AA', the foregoing rule applies to this condition also.

The departure of a course may be found in a similar manner. For instance, the departure of AB is equal to $OF - OE$; the departure of $A'B'$ is equal to $OF' - OE'$; and the departure of AA' is $OE - (-OE') = OE + OE'$.

When the reference axes pass through one end of a course, the total latitude and the total departure of this end are zero and the latitude and the departure of the course are equal, respectively, to the total latitude and the total departure of the other end of the course.

14. In the preceding article, the numerical values of the latitude and the departure of a course were found, but the direction of the course was not considered. This is determined by inspection of the given values for the ends of the course or by means of a sketch showing the relative positions of the ends.

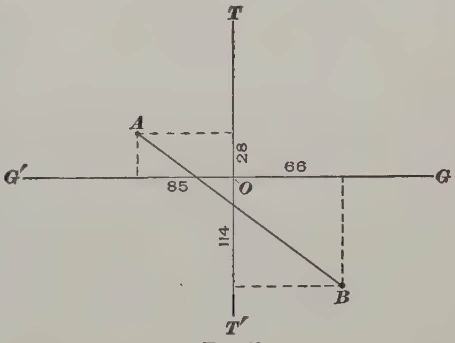


Fig. 10

Example 1.—In Fig. 10, the total latitude and the total departure of the point A are, respectively, 28 and -85; and, for the point B, the total latitude is -114 and the total departure 66. Find the latitude and the departure of the course AB.

Solution.—The difference between the total latitudes of A and B is $28-(-114)=28+114=142$. Since B is evidently south of A, the latitude of AB is a southing and is, therefore, -142. Ans.

The difference between the total departures of A and B is $66-(-85)=66+85=151$. In this case B is east of A and the departure of AB is an easting. Hence, its value is 151. Ans.

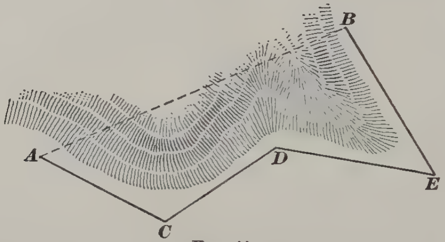


Fig. 11

Example 2.—In Fig. 11, A and B are two points on opposite sides of a hill, between which it is necessary to construct a tunnel for a railroad. In order to locate a straight line from A to B, the random survey $ACDEB$ was run around the hill. From the field notes given in the following table, compute the length and the azimuth of AB.

Course	Length	Azimuth
AC	122.14	121° 45′
CD	118.60	58° 10′
DE	120.48	103° 20′
EB	136.91	334° 3′

Solution.—First, the azimuths are changed to bearings and the latitudes and the departures of the courses are computed. The results are tabulated as follows:

Course	Length	Bearing	Latitude		Departure	
			N	S	E	W
AC	122.14	S 58° 15′ E		64.27	103.86	
CD	118.60	N 58° 10′ E	62.55		100.76	
DE	120.48	S 76° 40′ E		27.79	117.23	
EB	136.91	N 25° 57′ W	123.10			59.91

The reference axes are taken through A. The sum of the northings is $62.55+123.10=185.65$; of the southings, $64.27+27.79=92.06$; of the eastings, 103.86+100.76+117.23=321.85; and of the westings, 59.91. The difference between the northings and the southings is 185.65−92.06 =93.59; the total latitude of *B* is, therefore, +93.59. Similarly, the total departure of *B* is 321.85−59.91=261.94. Since the total latitude and the total departure of *A* are equal to zero, the latitude of *AB* is equal to the total latitude of *B* and the departure of *AB* is the total departure of *B*. Thus, the latitude t of *AB* is +93.59 ft. and its departure g is +261.94 ft. Then, by formula **3**, Art. **5**,

$$\tan G = \frac{g}{t} = \frac{261.94}{93.59}, \text{ whence } G = 70° \, 20'$$

Since both g and t are positive, the azimuth of *AB* is equal to G, or 70° 20′. Ans.

By formula **4**, Art. **5**,

$$l = \frac{g}{\sin G} = \frac{261.94}{\sin 70° \, 20'} = 278.16 \text{ ft.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. In the second and third columns of the following table are the lengths and the azimuths of the courses in the first column. (*a*) Compute the latitudes and the departures of the courses and compare the results with those given in the fourth and fifth columns. (*b*) Assuming the reference axes to pass through *A*, determine the total latitude and the total departure of each corner; the values are given in the last columns.

Course	Length	Azimuth	Latitude	Departure	Corner	Total Latitude	Total Departure
AB	638.85	35° 16′	+521.60	+368.86	*B*	+521.60	+ 368.86
BC	943.32	119° 43′	−467.62	+819.26	*C*	+ 53.98	+1,188.12
CD	719.98	171° 58′	−712.92	+100.62	*D*	−658.94	+1,288.74
DE	620.20	260° 38′	−100.94	−611.93	*E*	−759.88	+ 676.81
EF	649.07	352° 56′	+644.14	− 79.85	*F*	−115.74	+ 596.96

2. Determine (*a*) the length and (*b*) the azimuth of the line from *A* to *F* in example 1.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ 608.05\\(b)\ 100° \, 58'\end{array}\right.$$

3. The latitude of a course is −317 feet and its departure is 425 feet. Determine (*a*) the length and (*b*) the bearing of the line to the nearest five minutes.

$$\text{Ans.}\left\{\begin{array}{l}(a)\ 530 \text{ ft.}\\(b)\ \text{S } 53° \, 15' \text{ E}\end{array}\right.$$

4. Find the latitude t and the departure g of the line from *C* to *E* in example 1.

$$\text{Ans.}\left\{\begin{array}{l}t = -813.86\\g = -511.31\end{array}\right.$$

BALANCING SURVEYS

ERROR OF CLOSURE

15. Total Error of Closure.—If the lengths and the bearings of all the courses of a survey were measured with absolute exactness, and those measurements were plotted with equal accuracy, the courses of a survey that starts and ends at the same point would form a closed polygon. However, no matter how carefully a survey is made, all sources of error cannot be entirely eliminated. Therefore, if the courses are plotted according to the field measurements, the end of the survey never coincides exactly with the starting point. In Fig. 12, $ABCDEA_1$ is the plot of a closed traverse made according to the recorded field notes. The starting point A and the end A_1 represent the same point on the ground but do not coincide on the map because of errors in the work (the distance between A and A_1 is shown exaggerated for clearness). The length of the line joining the beginning of the first course of a closed traverse and the end of the last course determined by the field measurements is called the total error of closure. In Fig. 12, the length of AA_1 is the total error of closure.

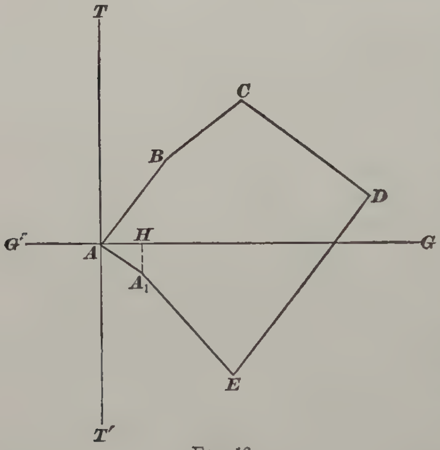


FIG. 12

16. Relative Error of Closure.—It is to be expected that in long surveys the total error of closure will be greater than in short surveys, if the same precautions are taken in both cases. Therefore, the total error of closure does not indicate the relative degrees of accuracy of two surveys of different lengths. As a basis for determining the degree of accuracy of a survey of any length, the error should be reduced to the amount in a unit length. This is found by dividing the total error of closure by the total length of the courses of the survey, and is called the relative error of closure. For example, if the sum of the lengths of the courses is 3,575 feet and the total error of closure is 2.5 feet, the relative error of closure is $\frac{2.5}{3{,}575}$, or $\frac{1}{1{,}430}$. The relative error of closure is commonly expressed as a fraction having a numerator equal to unity, or as a ratio such as 1 in 1,430, although it is sometimes given as a decimal, such as .0007. In an ordinary compass survey, the relative error of closure should be less than $\frac{1}{300}$, and in ordinary transit work it should be not greater than $\frac{1}{5{,}000}$. In some cases, the limit is $\frac{1}{10{,}000}$, and in very accurate work, the error should be less than $\frac{1}{20{,}000}$.

17. Formulas for Error of Closure.—In Fig. 12, let the line TT' represent the meridian through A, and GG' the east-and-west reference axis. Draw A_1H parallel to TT'. Then the total latitude of A_1, which is HA_1, is equal to the difference between the sum of the northings and the sum of the southings of the courses. The total departure of A_1, which is AH, is equal to the difference between the sum of the eastings and the sum of the westings.

From the right triangle AHA_1,

$$AA_1 = \sqrt{\overline{HA_1}^2 + \overline{AH}^2}$$

Let E = total error of closure;

S_t = difference between sums of northings and southings;

S_g = difference between sums of eastings and westings.

Then, $$E=\sqrt{S_t^2+\mathrm{S}_g^2} \qquad (1)$$

Also, let e = relative error of closure;

S_l = sum of lengths of courses.

Then, $$e=\frac{E}{S_l} \qquad (2)$$

In Fig. 12, the reference axes were taken through A but formula **1** is true for any positions of the axes.

18. Determination of Error of Closure.—The calculations for finding the error of closure from the notes in Table I will be given for illustration. The sum of the northings is 651.59 + 296.62 + 499.89 = 1,448.10; of the southings, 403.22 + 619.79 + 256.27 + 168.05 = 1,447.33; of the eastings, 101.45 + 847.02 + 436.20 + 103.53 = 1,488.20; and of the westings, 290.68 + 223.44 + 974.60 = 1,488.72. The difference between the northings and the southings is $S_t = 1{,}448.10 - 1{,}447.33 = 0.77$ foot; and the difference between the eastings and the westings is $S_g = 1{,}488.72 - 1{,}488.20 = 0.52$ foot. The total error of closure is, therefore,

$$E=\sqrt{S_t^2+S_g^2}=\sqrt{.77^2+.52^2}=.93 \text{ foot}$$

The sum of the lengths of the courses is $S_l = 4{,}703.33$, say 4,703, and the relative error of closure is

$$e=\frac{E}{S_l}=\frac{.93}{4{,}703}=.000198=\frac{1}{5{,}060}$$

19. In computing the error of closure for a survey, only the courses that form boundaries of the field should be considered; lines locating objects, or courses that were taken for reference, should be omitted. On the other hand, the courses considered must form a closed figure in the field.

If the error of closure is very small, say less than 1 in 20,000, it may be disregarded and the measurements assumed to be correct. If the error of closure indicates that the work is sufficiently accurate but not very exact, the field measurements are corrected to eliminate the error and thus balance the survey. If the error of closure is greater than the allowable value for the kind of work, the measurements and the calculations should be checked to locate any mistakes.

EXAMPLE FOR PRACTICE

The field notes and the calculated latitudes and departures for a survey follow. Determine (*a*) the total error of closure and (*b*) the relative error of closure.

Course	Length	Azimuth	Latitude	Departure
AB	638.85	35° 16′	+521.60	+368.86
BC	943.32	119° 43′	−467.62	+819.26
CD	719.98	171° 58′	−712.92	+100.62
DE	620.20	260° 38′	−100.94	−611.93
EF	649.07	352° 56′	+644.14	− 79.85
FA	607.36	280° 57′	+115.37	−596.30

Ans. { (*a*) .76
(*b*) .00018

METHODS OF BALANCING

20. Explanation.—In Fig. 12, A and A_1 represent the same point on the ground. Hence, in a closed traverse the sum of the northings of the courses should equal the sum of the southings, and the sum of the eastings should equal the sum of the westings. In order to make each of the values, S_l and S_d, in the formula $E=\sqrt{S_l^2+S_d^2}$ equal to zero, and thus reduce the error of closure to zero, it is necessary to revise the notes. The adjustment, which consists of applying corrections to the latitudes and the departures of the courses, is called *balancing the survey*, although it might be called *balancing the notes*.

21. Correcting Measurements.—There are two general methods of correcting the measurements of the courses. One is based on the fact that the chances of error are greater in making a difficult measurement than in making one under more favorable conditions. In the other method, it is assumed that all lines are measured under similar conditions. For ordinary surveys, the second method is usually preferred, as it is considered an unnecessary refinement to apply the first method.

22. Transit and Compass Methods.—In a transit survey, it is commonly assumed that the instrument work and the chaining are done with equal accuracy, but in a compass survey the angular error is necessarily larger than that permissible in transit work. Therefore, the method of balancing for a transit survey differs from that for a compass survey.

23. Angular Error.—In a transit survey, it is customary to test the accuracy of the instrument work before an attempt is made to balance the survey. The error in the measurement of the angles of a closed traverse is determined as follows:

If the method of direct angles is employed, the measured angles are the interior angles of a polygon, and, therefore, their sum should be equal to $180° \times (n-2)$, in which n is the number of courses in the traverse. Thus, the sum of the interior angles of a field having six sides should be $180° \times (6-2) = 720°$. If the sum of the measured angles is $720°\ 1'$, the angular error is 1 minute or 60 seconds.

If the method of deflection angles is used, the difference between the sum of the deflections to the right and the sum of the deflections to the left should be equal to 360°. The amount by which the actual difference varies from 360° is the angular error.

If the method of azimuths is used, the azimuth of one course is found both at the beginning and at the end of the survey; the second value should agree with the first. Thus, in Fig. 7, the azimuth of AD may be determined from A when the survey is started and then the azimuth of DA may be found when the transit is set at D. The back azimuth of DA should be equal to the azimuth of AD; the difference between the two values is the angular error.

The allowable angular error is generally assumed to be proportional to the square root of the number of angles measured.

Let E_a = total allowable angular error;
N = number of angles measured;
e_a = allowable error in one angle.

Then,
$$E_a = e_a \sqrt{N}$$

The value of e_a may be taken as about one-half of the least reading of the transit vernier.

If the actual angular error is not much greater than the allowable error, the transit work may be assumed to be correct. Nevertheless, some of the angles are corrected so that no angular error remains. The angles where errors probably occurred can usually be determined by inspection of the survey. In other cases, the errors should be assumed to be in the angles adjacent to short lines, since any error in sighting or setting up causes a greater angular error when the sides of the angle are short than when they are long.

If the actual error exceeds the total allowable error by more than 1 or 2 minutes, the angles must be measured again to get more accurate results. It is, therefore, advisable to test the accuracy of the transit work while the party is yet in the field.

24. Balancing Transit Surveys.—After the angles of a transit survey have been adjusted, the latitudes and the departures of the courses are corrected. The corrections to be applied are determined by the following formulas:

$$c_t = t \times \frac{S_t}{R_t} \qquad (1)$$

$$c_g = g \times \frac{S_g}{R_g} \qquad (2)$$

in which t = latitude of course;

c_t = correction to latitude;

S_t = difference between sum of northings and sum of southings of courses of survey;

R_t = sum of numerical values of northings and southings;

g = departure of course;

c_g = correction to departure;

S_g = difference between sum of eastings and sum of westings;

R_g = sum of numerical values of eastings and westings.

If the northings exceed the southings, the corrections are to be subtracted from the north latitudes and added to the south latitudes; if the southings are greater, the corrections are added to the north latitudes and subtracted from the south latitudes. Similarly, if the eastings are greater than the westings, the corrections are subtracted from the east departures and added to the west departures; if the westings are greater, the corrections are added to the east departures and subtracted from the west departures.

The lengths and the azimuths of the courses to be used in the description of the survey are calculated from the corrected latitudes and departures. Occasionally, the azimuth of a course is changed again by the changes in the latitude and the departure but, usually, the direction of the line is not affected.

Let $g' =$ corrected departure;
$t' =$ corrected latitude;
$G' =$ corrected bearing;
$l' =$ corrected length.

Then,

$$\tan G' = \frac{g'}{t'} \qquad (3)$$

$$l' = \frac{g'}{\sin G'} \qquad (4)$$

$$l' = \frac{t'}{\cos G'} \qquad (5)$$

$$l' = \sqrt{t'^2 + g'^2} \qquad (6)$$

The azimuth can be found from the bearing.

25. In Table II are shown the calculations for balancing the transit survey for which the notes are given in Table I. It is assumed that the transit work is correct. The azimuths and the lengths of the courses, without parentheses, are copied from the field notes, and the latitudes and the departures, without parentheses, are calculated from them. The difference between the northings and the southings is $S_t = .77$ and their sum is $R_t = 2,895.43$. Similarly, $S_g = .52$ and $R_g = 2,976.92$. The relative error of closure was found to be $\frac{1}{5,060}$; therefore, the work is sufficiently accurate for balancing the survey.

Next, the corrections to the latitudes and departures are computed by means of formulas **1** and **2**, Art. **24**. Since the northings exceed the southings, the corrections will be subtracted from the north latitudes and added to the south latitudes; since the westings are greater than the eastings, the corrections will be subtracted from the west departures and added to the east departures. The corrections are tabulated as follows, $\frac{S_t}{R_t}$ being equal to $\frac{.77}{2{,}895}$, or .00027, and $\frac{S_g}{R_g}$ being equal to $\frac{.52}{2{,}977}$, or .00017. Since the method of balancing is approximate, the values of t and g may be taken to the nearest 10 feet. A slide rule will prove convenient for these computations.

COURSE	c_t	c_g
AB	$650 \times .00027 = -.17$	$100 \times .00017 = +.02$
BC	$300 \times .00027 = -.08$	$850 \times .00017 = +.15$
CD	$400 \times .00027 = +.11$	$440 \times .00017 = +.07$
DE	$620 \times .00027 = +.17$	$100 \times .00017 = +.02$
EF	$260 \times .00027 = +.07$	$290 \times .00017 = -.05$
FG	$500 \times .00027 = -.13$	$220 \times .00017 = -.04$
GA	$170 \times .00027 = +.04$	$970 \times .00017 = -.17$

In order to make the error of closure zero, the sum of the numerical values of the corrections to the latitudes must equal S_t and the sum of the corrections to the departures must equal S_g. For this reason, the computed values are adjusted slightly; thus, c_t for AB is called .17 although it is a little nearer .18, c_t for GA is taken as .04, c_g for BC is made .15, and c_g for GA is called $-.17$. The corrected latitudes and departures are written in parentheses above the calculated values. In practice, the calculated values are crossed out, but not erased, and no parentheses are used around the corrected values.

The corrected lengths and azimuths are then determined from the corrected latitudes and departures. For instance, the corrected angle G' for BC is found by the relation, $\tan G' = \frac{g'}{t'} = \frac{847.17}{296.54}$, whence $G' = 70° 42'$. Since g' and t' are both positive, the azimuth of *BC* is 70° 42′. The corrected length of *BC* is

$$l' = \frac{g'}{\sin G'} = \frac{847.17}{\sin 70^\circ\ 42'} = 897.62 \text{ feet.}$$

It will be noticed that in some cases the corrected lengths are greater than the corresponding measured distances. Since lengths are likely to be measured too long rather than too short, unless the tape is too long, some engineers prefer to make the adjustments entirely by subtracting from the greater values so that no lengths will be increased.

TABLE II

COMPUTATIONS FOR BALANCING TRANSIT SURVEY

Course	Azimuth	Length	Latitude		Departure	
			N	S	E	W
		(659.27)	(651.42)		(101.47)	
AB	8° 51′	659.43	651.59		101.45	
		(897.62)	(296.54)		(847.17)	
BC	70° 42′	897.46	296.62		847.02	
		(594.11)		(403.33)	(436.27)	
CD	132° 45′	594.01		403.22	436.20	
		(628.55)		(619.96)	(103.55)	
DE	170° 31′	628.37		619.79	103.53	
	(228° 35′)	(387.56)		(256.34)		(290.63)
EF	228° 36′	387.52		256.27		290.68
		(547.41)	(499.76)			(223.40)
FG	335° 55′	547.55	499.89			223.44
		(988.80)		(168.09)		(974.43)
GA	260° 13′	988.99		168.05		974.60
		4,703.33	1,448.10	1,447.33	1,488.20	1,488.72
			1,447.33	1,448.10	1,488.72	1,488.20
			.77	2,895.43	2,976.92	.52

26. Balancing Compass Surveys.—The method of balancing a compass survey differs from that employed for a transit survey only in the formulas for determining the corrections which are applied to the latitudes and the departures. For a compass survey, the following formulas are used:

Let l = length of course;
c_t = correction to latitude;
c_g = correction to departure;
S_l = sum of lengths of courses;
S_t = difference between sum of northings and sum of southings;
S_g = difference between sum of eastings and sum of westings

Then,
$$c_t = l \times \frac{S_t}{S_l} \qquad (1)$$

$$c_g = l \times \frac{S_g}{S_l} \qquad (2)$$

Since the bearings in a compass survey are accurate only to the nearest 5 minutes, they are seldom affected by the adjustment of the latitudes and the departures. The lengths of the courses, however, must usually be changed to agree with the corrected latitudes and departures. The new bearings may be determined by applying formula **3**, Art. **24**. The new lengths may be found by formula **4**, **5**, or **6**, Art. **24**, but they can be determined more easily by applying corrections to the measured lengths. The corrections may be calculated by means of the formula

$$c_l = t \times \frac{S_t}{S_l} + g \times \frac{S_g}{S_l} \qquad (3)$$

in which c_l = correction to length of course;
t = latitude of course;
g = departure of course;

S_l, S_t, and S_g have same meanings as in formulas **1** and **2**.

It will be noticed that the signs of all the terms in the formula are plus, but that is the case only when the corrections are added to both the latitude and the departure of the course in question. The method of applying the formula under other conditions is shown in the following calculations.

27. In the first three columns of Table III are given the courses, the bearings, and the lengths (without parentheses), which are copied from the field notes for a compass survey. In the other columns are the calculated latitudes and departures (also without parentheses). It is required to balance the survey. The corrections to the latitudes and the departures are found by formulas **1** and **2**, Art. **26.** Here, the sum of the northings is 545.1, the sum of the southings is 547.8, and S_t is $547.8-545.1=2.7$. The sum of the eastings is 441.1, the sum of the westings is 439.2, and S_g is $441.1-439.2=1.9$. The sum of the lengths, S_l, is 1,614.3. Then, $\frac{S_t}{S_l}=\frac{2.7}{1{,}614}=.0017$ and $\frac{S_g}{S_l}=\frac{1.9}{1{,}614}=.0012$. The computations may be arranged as follows:

COURSE	c_t	c_g
AB	$300\times.0017=.5$	$300\times.0012=-.4$
BC	$220\times.0017=.4$	$220\times.0012=-.3$
CD	$190\times.0017=-.3$	$190\times.0012=-.2$
DE	$270\times.0017=-.5$	$270\times.0012=.3$
EF	$370\times.0017=-.6$	$370\times.0012=.4$
FA	$250\times.0017=.4$	$250\times.0012=.3$

Since the southings exceed the northings, the corrections are added to the north latitudes and subtracted from the south latitudes, as indicated by the signs. Similarly, since the eastings exceed the westings, the corrections are added to the west departures and subtracted from the east departures. Here the corrected values are written in parentheses above the computed values; but in practice the computed values would be crossed out and the corrected values would not be enclosed in parentheses.

The corrections to the original lengths are found by using formula **3**, Art. **26.** For *AB*, $c_l=260\times.0017-150\times.0012=.3$. Since the correction was added to the latitude of *AB*, the term $t\times\frac{S_t}{S_l}$ is plus; but since the correction was subtracted from the departure of *AB*, the term $g\times\frac{S_g}{S_l}$ is minus. The value of c_l is plus and is added to the original length of *AB*

For *BC*, $c_l = 37 \times .0017 - 220 \times .0012 = -.2$. The value of c_l is minus and is subtracted from the original length of *BC*.

For *CD*, $c_l = -180 \times .0017 - 72 \times .0012 = -.4$. The corrections were subtracted from both the latitude and the departure of *CD*; hence, both $t \times \frac{S_t}{S_l}$ and $g \times \frac{S_g}{S_l}$ are minus.

For *DE*, $c_l = -270 \times .0017 + 58 \times .0012 = -.4.$
For *EF*, $c_l = -100 \times .0017 + 360 \times .0012 = .3.$
For *FA*, $c_l = 250 \times .0017 + 20 \times .0012 = .4.$

TABLE III

COMPUTATIONS FOR BALANCING COMPASS SURVEY

Course	Bearing	Length	Latitude		Departure	
			N	S	E	W
	(N 30° 15′ E)	(300.3)	(259.4)		(151.1)	
AB	N 30° 20′ E	300.0	258.9		151.5	
	(N 80° 15′ E)	(220.4)	(37.4)		(217.2)	
BC	N 80° 20′ E	220.6	37.0		217.5	
		(193.4)		(179.6)	(71.9)	
CD	S 21° 50′ E	193.8		179.9	72.1	
	(S 12° 15′ W)	(274.6)		(268.3)		(58.3)
DE	S 12° 10′ W	275.0		268.8		58.0
	(S 74° 45′ W)	(375.2)		(98.5)		(362.0)
EF	S 74° 40′ W	374.9		99.1		361.6
		(250.4)	(249.6)			(19.9)
FA	N 4° 30′ W	250.0	249.2			19.6
		1,614.3	545.1	547.8	441.1	439.2
				545.1	439.2	
				2.7	1.9	

Approximate values of *t* and *g* are close enough for these calculations. The corrected lengths and bearings are written here in parentheses over the original values in the table.

28. Supplying Omissions.—It is sometimes impossible to determine by direct measurement the lengths and the directions of all the sides of a closed field; moreover, omissions sometimes occur in the notes from accident. In a closed survey in which there are two omissions, the missing parts can be supplied from the other parts by calculation. Suppose, for example, that very thick woods make it impracticable to measure the length and the azimuth of the side CD in the survey for which the notes are given in Table I. Then D would be taken as the starting point of the survey and the notes would be arranged in the following form:

Course	Azimuth	Length	Latitude		Departure	
			N	S	E	W
DE	170° 31′	628.37		619.79	103.53	
EF	228° 36′	387.52		256.27		290.68
FG	335° 55′	547.55	499.89			223.44
GA	260° 13′	988.99		168.05		974.60
AB	8° 51′	659.43	651.59		101.45	
BC	70° 42′	897.46	296.62		847.02	

The length and the azimuth of CD may be found from the total latitude and the total departure of C with respect to axes through D. The sum of the northings is $499.89+651.59+296.62=1{,}448.10$; of the southings, $619.79+256.27+168.05=1{,}044.11$; of the eastings, $103.53+101.45+847.02=1{,}052.00$; and of the westings, $290.68+223.44+974.60=1{,}488.72$. The total latitude of C with respect to D is $1{,}448.10-1{,}044.11=403.99$, and its total departure is $1{,}488.72-1{,}052.00=436.72$. Since C is north of D, the latitude of CD is a southing; and since C is west of D, the departure of CD is an easting. Then, by formula **3**. Art. **5**,

$$\tan G=\frac{g}{t}=\frac{436.72}{403.99}$$

Whence $$G=47°\ 14'$$

Since g is positive and t is negative, the azimuth of CD is equal to $180°-G=180°-47°\ 14'=132°\ 46'$. The length of CD is, by formula **4**, Art. **5**,

$$l=\frac{g}{\sin G}=\frac{436.72}{\sin 47°\ 14'}=594.89 \text{ feet}$$

The case shown in Fig. 11 is another in which it is necessary to supply omissions. The line *AB* may be considered as a side of the closed survey *ACDEBA*; and its length and azimuth may be found from the measurements of the other sides as previously shown.

The surveyor should make every measurement that is practicable so as to avoid the necessity of supplying omissions by computation; for, when values are determined in this manner, it must be assumed that the remaining field notes are exactly correct. Consequently, there are no means of balancing the survey, and all errors are thrown in the part or parts supplied.

EXAMPLES FOR PRACTICE

1. In the following table are given the latitudes and the departures of the courses of a closed transit traverse as computed from the field measurements. (*a*) Calculate the total error of closure. (*b*) Verify the corrected latitudes and departures and the corrected azimuths and lengths.

Ans. (*a*) .70

Course	Calculated Latitude	Calculated Departure	Corrected Latitude	Corrected Departure	Corrected Azimuth	Corrected Length
AB	+521.60	+368.86	+521.51	+368.94	35° 17′	638.87
BC	−467.62	+819.26	−467.70	+819.43	119° 43′	943.54
CD	−712.92	+100.62	−713.04	+100.64	171° 58′	720.10
DE	−100.94	−611.93	−100.96	−611.80	260° 38′	620.07
EF	+644.14	− 79.85	+644.03	− 79.84	352° 56′	648.97
FA	+116.18	−597.50	+116.16	−597.37	281° 00′	608.56

2. The following table contains the lengths and the bearings of the courses of a compass survey, the values of the latitudes and the departures as computed from the original notes, and the corrected values of the lati-

Course	Bearing	Length Chains	t	g	Corrected t	Corrected g	Corrected Length
AB	N 41° 30′ E	10.47	+ 7.84	+ 6.94	+ 7.85	+ 6.96	10.49
BC	N 75° 15′ E	11.86	+ 3.02	+11.47	+ 3.03	+11.49	11.89
CD	S 20° 45′ E	11.64	−10.88	+ 4.12	−10.87	+ 4.14	11.64
DE	S 57° 45′ W	15.78	− 8.42	−13.35	− 8.40	−13.32	15.74
EA	N 48° 00′ W	12.52	+ 8.38	− 9.30	+ 8.39	− 9.27	12.51

tudes, the departures, and the lengths. (*a*) Verify the original values of the latitudes and the departures, and the corrected values of the latitudes, the departures, and the lengths. (*b*) Calculate the relative error of closure.

Ans. (*b*) .002

3. From the following notes, determine (*a*) the length and (*b*) the bearing of the line *PQ*.

Ans. (*a*) 823.0 ft.
(*b*) N 74° 5′ E

Course	Length	Bearing
PA	300.0	N 45° E
AB	200.0	S 70° E
BC	250.0	due east
CQ	163.4	N 60° E

PLOTTING SURVEYS

INTRODUCTION

29. Definition and Methods.—A plot of a survey shows on paper the relative locations of points and objects on the ground. The plot is constructed by drawing lines to represent the courses whose lengths and directions have been determined by measurement and have been recorded in the notes. Although there are many methods of plotting, the points of a survey are commonly located either by latitudes and departures, or by lengths and bearings or azimuths. In a compass survey, the method of lengths and bearings is always used. In a transit survey, the method of latitudes and departures is generally employed for locating important points and the method of lengths and azimuths is used for filling in details.

30. Instruments.—The instruments commonly used in plotting are a drawing board, a sharp hard pencil, a scale, a protractor, two triangles, and a **T** square, all of which have been described in connection with *Geometrical Drawing*. Special forms of the **T** square and the protractor will be described here. A long straightedge, preferably of steel, is sometimes convenient. A *parallel ruler* is often used instead of a **T** square.

31. T Square.—In one style of **T** square, the blade is fixed at right angles to the head, and, therefore, only parallel lines at right angles to the edge of the board can be drawn. For drawing parallel lines not perpendicular to the edge of the board, the type of **T** square shown in Fig. 13 is convenient.

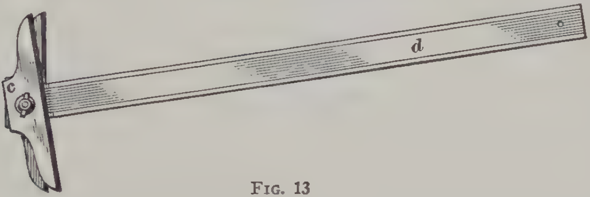


Fig. 13

The head is divided into two separate parts, which are held together by means of a bolt and a nut.

When the nut is loosened, one part can be rotated to the right or to the left with respect to the other section; the two pieces are then held in position by tightening the nut. The upper part of the head, *c*, is rigidly fixed at right angles to the blade *d*, and, when the lower part of the head is kept against the edge of the drawing board, the blade moves parallel to itself as the instrument slides.

32. Parallel Ruler.—The type of parallel ruler generally used is shown in Fig. 14. It consists of a metal straightedge which is mounted on milled rollers of equal diameter attached to a common shaft. If the ruler is carefully rolled over a surface without lifting either roller, the straightedge will move parallel to itself. Hence, if one edge of the ruler is placed along a given line, a parallel line through any point can be obtained by rolling the instrument until an edge passes through the desired point, and then drawing a line along that edge. Owing to the greater cost of parallel rulers and to the fact that

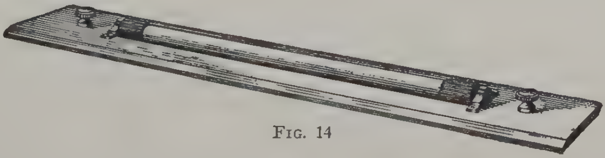


Fig. 14

more care must be exercised to give as good results, parallel rulers are not used so much as **T** squares.

33. Protractors.—The simple semicircular protractor has been described in a preceding Section. The type shown in Fig. 15 is an ordinary metal protractor with a rotating arm, or blade, *a*, which extends from the center of the graduated circle. In order that the center may be set accurately at the desired point on the paper, it is marked by cross lines on a transparent celluloid disc in the joint *b* by which the blade is attached. For laying off angles in transit work, the arm of the protractor is supplied with a vernier *c*, by means of which settings can be made to the nearest minute. In Fig. 15 the half-degree graduations are omitted to make the illustration clearer.

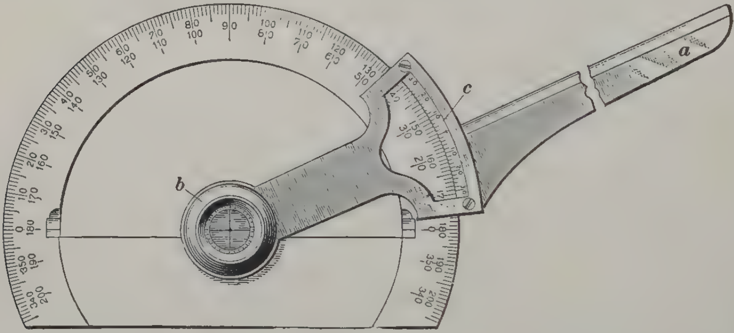


Fig. 15

In using protractors such as those already described, it is necessary to move the protractor for every new point where angles are measured; thus, much time is required and there is likely to be some error in placing the protractor in the proper position at each point. The form of protractor often used in plotting lines by bearings or azimuths is the paper protractor, which can be tacked to the drawing board and kept in one position for plotting many lines. The *paper protractor* is a full circle printed from accurately engraved plates on a sheet of drawing paper, tracing paper, or bristol board. The circle is from 8 to 14 inches in diameter and is graduated to half or quarter degrees; the numbers of the graduations are not printed on the sheets but are written on by the draftsman in the manner most convenient for his purpose. For bearings, the graduations are numbered in quadrants from 0° to 90° in each direction, similar to those on the needle circle of a compass, so that bearings can be readily found. For azimuths the numbers increase from 0 to 360 clockwise, as on the horizontal limb of a transit. The center of the graduated circle is marked by a cross.

PLOTTING BY LENGTHS AND BEARINGS

34. Preliminaries to Plotting.—Before plotting is started, it is necessary to select the scale, the direction of the meridian, and the location of the first point of the survey, so that the plot will not run off the paper. It is a slight advantage in plotting with a **T** square to have the meridian parallel to the edge of the paper, but there is no objection to having the meridian in any other direction. However, north should always be toward the top of the paper. The best scale and the location of the first point can be determined from the latitudes and departures, or by careful examination of the field notes and a rough sketch of the survey.

In ordinary work the scale used is so small that the corrections to the measurements that are required to balance the survey do not affect the plotting. Hence, the plot can be drawn from the measurements in the notes, and it is not necessary to balance the survey before plotting. A considerable error of closure in the plot indicates a mistake either in the field measurements or in the plotting, and should be investigated.

35. Plotting Bearings With Paper Protractor.—When a paper protractor is used, a long line representing the direction of the meridian is drawn in a convenient position on the paper, and at any place along this line the protractor is tacked to the board so that the zero points coincide with the line. In Fig. 16, the meridian is indicated by the line with a half arrowhead. The starting point *A* of the survey is marked by a pinhole with a small circle around it. For making this hole, a needle inserted in a small piece of wood as a handle is very convenient. If a parallel ruler is used, one edge of it is set on the protractor to read the bearing of the first course. In Fig. 16, the bearing of *A B* is N 40° E and the ruler is, therefore, placed in the position *a a*. Then, it is rolled to the position *a′a′*, where the edge passes through *A*, and an indefinite line is drawn. On this line, the length of the course *A B* is laid off to the proper scale, and the point *B* is located and marked with a pinhole and a circle. The parallel ruler does not have to extend across both parts of the graduated circle, but may be set to pass through the center mark and the proper graduation in one quadrant. Next the ruler is set on the protractor in the position *b b* to read the bearing of *BC*, which is N 70° E, and is rolled to the position *b′b′* so that the edge passes through *B*. An indefinite line is drawn along the edge and, on this line, the length of *BC* is laid off to scale to locate *C*. Point *D* is located in a similar manner. The ruler is first placed in the position *c c* to read a bearing of S 30° E, and is then rolled to the position *c′c′* to pass through *C*; the distance *CD* is laid off in the direction of the ruler.

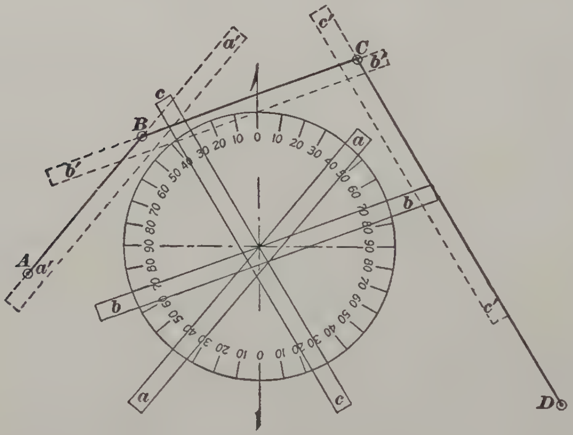


FIG. 16

In case it becomes impossible to transfer a direction from the protractor to the required position of the line by means of the parallel ruler, the protractor can be shifted to any other position on the paper. When the protractor is to be moved, a line is drawn parallel to the meridian, and the zero points of the protractor are set on this line.

If no parallel ruler is at hand, the directions of the lines can be transferred from the protractor to the required points by means of an adjustable T square, or better, by means of two triangles used for drawing parallel lines as explained elsewhere. If the triangles have long edges, as 12 or 14 inches, the direction of a line can be transferred with great rapidity after a little practice; of course, smaller triangles can also be used but they require more shifting than the larger ones. Azimuths can be laid off with a protractor in the same way as bearings

36. Plotting With Other Protractors.

When an angle is to be laid off with a metal or celluloid protractor, the center of the protractor must be set over the vertex of the angle. Therefore, the protractor must be placed at each point over which the transit was set in the field.

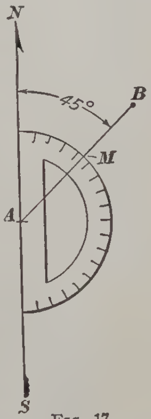


FIG. 17

The direction of the meridian is marked by a long line at some convenient position on the paper and the starting point of the survey is located. If a semicircular protractor without a rotating arm is used, a meridian is drawn through the starting point and the protractor is set with its center at that point and its zero graduations on the meridian. If the bearing of the line is northeast or southeast, the protractor is placed to the right of the meridian, and if the bearing is northwest or southwest, the protractor is placed to the left of the meridian. In Fig. 17, A is the starting point and NS is the meridian. Then a point is marked at the edge of the protractor opposite the graduation corresponding to the bearing of the first course. In Fig. 17, the bearing of AB is taken as N 45° E and the point M is marked at the edge of the protractor opposite the

45° graduation to the east from the north. The protractor is moved out of the way and an indefinite line is drawn through *A* and *M*. The length of *AB* is laid off to scale on this line and the point *B* of the survey is located and marked. A meridian is then drawn through the point *B*, and the operations are repeated to locate the next point of the survey.

If a semicircular protractor with an arm is used, the bearings can be laid off in less time and with less trouble. There is no necessity of drawing the meridian through each station since the straight part of the protractor can be placed against the edge of the **T** square, parallel ruler, or triangle, which is parallel to the meridian. By shifting the **T** square and by moving the protractor along the edge of the **T** square, the center of the protractor can be brought over the desired point. Then the blade of the protractor is set at the required bearing, and an indefinite line is drawn along the edge of the blade. The length of the course is laid off on this line to locate the next point of the survey. When only one bearing is laid off from a point, it is probably better to set the blade of the protractor at the desired bearing first, and then to shift the protractor to the proper position by bringing any part of the blade to the station from which the bearing is measured.

PLOTTING BY LATITUDES AND DEPARTURES

37. Introduction.—The first step in plotting by latitudes and departures is to select the scale of the map, the direction of the meridian, and the location of the starting point on the paper. Then the reference axes, to which the total latitudes and the total departures are referred, are drawn parallel and perpendicular to the meridian. In order to facilitate plotting, the paper is divided into a number of squares by drawing lines parallel to the axes and exactly 10 inches apart in both directions. These lines should be drawn very carefully, and the lengths of the diagonals of each square should be measured as a a check to see that they are 14.14 inches long. Then, points may be located by measuring from the nearest line rather than from a reference axis. For instance, if the scale of the map is

1 inch to 100 feet, the distance between the lines represents 10×100, or 1,000, feet. A point whose total departure is 2,300 feet may, therefore, be located by measuring 300 feet from the 2,000-foot line instead of 2,300 feet from the meridian.

38. Locating Points.—The values in Table IV are the total latitudes and the total departures of the points of the traverse for which the notes are given in Table I. The reference axes are assumed through *A*, and the corrected latitudes and departures taken from Table II are used.

The plot of the points by total latitudes and total departures is shown in Fig. 18. Station *B* is found by measuring 651.42 feet from *A* along the meridian to *B′* and then 101.47 feet from *B′* at right angles to the meridian. Point *C* is located by measuring 947.96 feet from *A* to *C′* and then 948.64 feet from *C′* at right angles to the meridian or parallel to the east-and-west axis. Point *C′* should not be located by measuring 296.54 feet from *B′*, because any mistake in *B′* would also be carried to *C′*. All distances should be measured from one of the carefully drawn lines that divide the paper into 10-inch squares. In locating a point, it is always convenient to measure the longer distance along one of these lines and the shorter distance from that line. Thus, point *E* can be located best by measuring along the east-and-west axis 488.46 feet from the auxiliary line 1,000 feet east of the meridian to *E″*, and then from *E″*.

TABLE IV

TOTAL LATITUDES AND DEPARTURES

Station	Total Latitude	Total Departure
A	0	0
B	+651.42	+ 101.47
C	+947.96	+ 948.64
D	+544.63	+1,384.91
E	− 75.33	+1,488.46
F	−331.67	+1,197.83
G	+168.09	+ 974.43

75.33 feet at right angles to the east-and-west axis. The work can be checked by measuring the lengths of the courses.

After the important points have been located by latitudes and departures, the details can be filled in by means of a protractor. The complete plot of the survey from the notes in

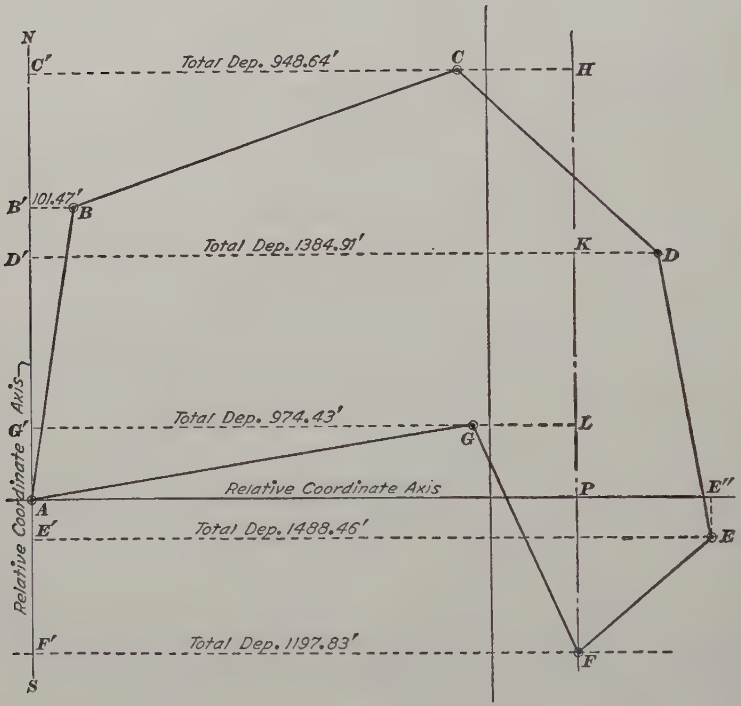
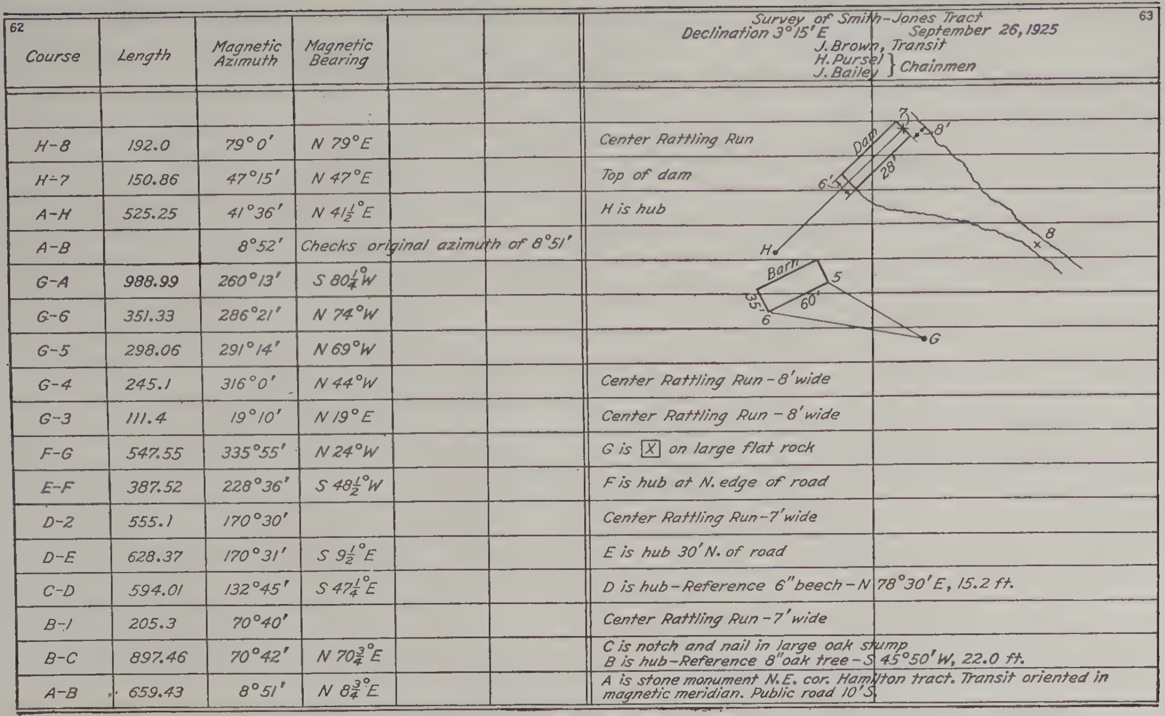


Fig. 18

Fig. 19 is shown in Fig. 20. The directions of lines on a map are generally given by bearings, even though azimuths were measured in the field. The numbers of the points locating objects would not be shown on the finished map but are given here for reference. In some surveys, the lengths and the bearings of the courses would not be shown either. When they are written on the map, however, the corrected values, and not the measured ones, should be used. Thus, the corrected values taken from Table II are used in Fig. 20.



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Survey of Smith-Jones Tract
Declination 3°15′E
September 26, 1925
J. Brown, Transit
H. Pursel } Chainmen
J. Bailey

Course	Length	Magnetic Azimuth	Magnetic Bearing			
H-8	192.0	79°0′	N 79°E			Center Rattling Run
H-7	150.86	47°15′	N 47°E			Top of dam
A-H	525.25	41°36′	N $41\frac{1}{2}$°E			H is hub
A-B		8°52′	Checks original azimuth of 8°51′			
G-A	988.99	260°13′	S $80\frac{1}{4}$°W			
G-6	351.33	286°21′	N 74°W			
G-5	298.06	291°14′	N 69°W			
G-4	245.1	316°0′	N 44°W			Center Rattling Run - 8′ wide
G-3	111.4	19°10′	N 19°E			Center Rattling Run - 8′ wide
F-G	547.55	335°55′	N 24°W			G is ☒ on large flat rock
E-F	387.52	228°36′	S $48\frac{1}{2}$°W			F is hub at N. edge of road
D-2	555.1	170°30′				Center Rattling Run - 7′ wide
D-E	628.37	170°31′	S $9\frac{1}{2}$°E			E is hub 30′ N. of road
C-D	594.01	132°45′	S $47\frac{1}{4}$°E			D is hub - Reference 6″ beech - N 78°30′E, 15.2 ft.
B-1	205.3	70°40′				Center Rattling Run - 7′ wide
B-C	897.46	70°42′	N $70\frac{3}{4}$°E			C is notch and nail in large oak stump B is hub - Reference 8″ oak tree - S 45°50′W, 22.0 ft.
A-B	659.43	8°51′	N $8\frac{3}{4}$°E			A is stone monument N.E. cor. Hamilton tract. Transit oriented in magnetic meridian. Public road 10′ S.

FIG. 19

CALCULATING AREAS

DOUBLE MERIDIAN DISTANCES

39. Explanation.—The most common method of calculating the area of a field bounded by straight lines is known as the double-meridian-distance method. The *double meridian distance* of a course, abbreviated *D. M. D.*, is equal to the sum of the total departures of the extremities of the course. For example, in Fig. 18, the double meridian distance of CD is equal to $C'C+D'D$, and that of AB is $B'B+0$, or $B'B$. Just

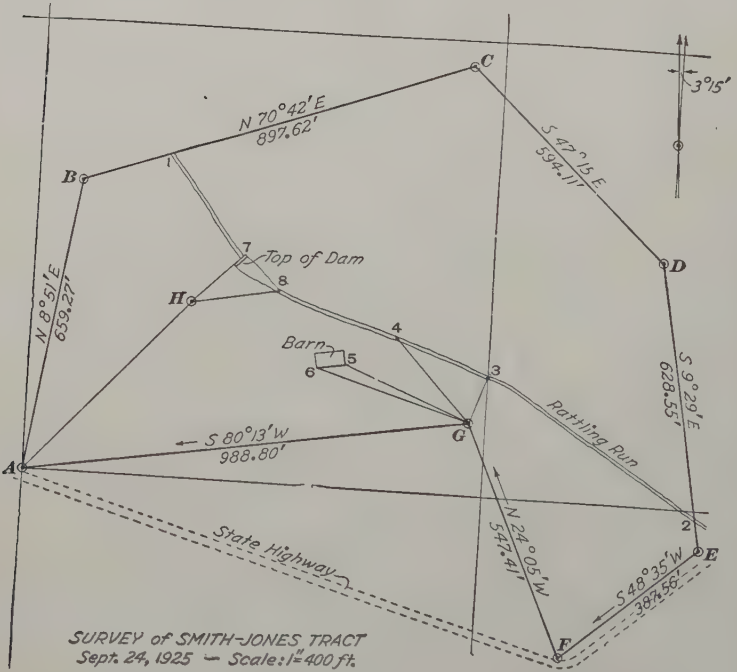

FIG. 20

as the total departures of the points of a traverse can be measured from any axis, the double meridian distances of the courses can be referred to any convenient meridian. If, in Fig. 18, the reference meridian is taken through F, the total departure of C is $-CH$ and the total departure of D is $+KD$. Hence, the double meridian distance of CD is equal to $KD-CH$. The numerical value of the double meridian distance of GA is equal to $GL+AP$; but, since the total departures of G and A are both negative, the double meridian distance of GA must be taken as $-(GL+AP)$. In order to make the calculations of double meridian distances easier and to avoid negative values, the reference meridian is usually taken through the most westerly corner of the field.

If the field has been plotted, the most westerly corner is readily selected. Sometimes, however, a study of the departures of the courses is sufficient to determine the proper corner. Occasionally, it may be necessary to assume a point as the most westerly and to calculate the total departures of the other corners from it. If all the values come out positive, the assumption is correct; otherwise, the corner having the greatest west total departure is the one to be selected. In Fig. 18, A is the most westerly corner.

40. Calculation of Double Meridian Distances.—The double meridian distances of the courses of a traverse may be calculated by determining the total departures of the corners and taking the sum of the total departures of the extremities of each course as explained in the preceding article. The survey should be balanced before the area is computed, and the corrected departures should be used in calculating the double meridian distances. If, in Fig. 18, the reference meridian is taken through A, the double meridian distance of BC, which is equal to the departure of B plus the departure of C, equals $101.47+948.64=1{,}050.11$.

The calculations may be made more easily by means of the following rule:

Rule.—*The double meridian distance of any course is equal to the sum of the double meridian distance of the preceding course,* *the departure of the preceding course, and the departure of the course itself. East departures are added and west departures are subtracted.*

In this method the courses must be taken in order, starting from the reference meridian. The double meridian distance of

Calculations for Determining Double Meridian Distances

First Method		Second Method	
Total Dep. *A*	= 0		
Total Dep. *B*	= 101.47		
D. M. D. *AB*	= 101.47	D. M. D. *AB* =	101.47 = Dep. *AB*
Total Dep. *B*	= 101.47	Dep. *AB* =	101.47
Total Dep. *C*	= 948.64	Dep. *BC* =	847.17
D. M. D. *BC*	= 1,050.11	D. M. D. *BC* =	1,050.11
Total Dep. *C*	= 948.64	Dep. *BC* =	847.17
Total Dep. *D*	= 1,384.91	Dep. *CD* =	436.27
D. M. D. *CD*	= 2,333.55	D. M. D. *CD* =	2,333.55
Total Dep. *D*	= 1,384.91	Dep. *CD* =	436.27
Total Dep. *E*	= 1,488.46	Dep. *DE* =	103.55
D. M. D. *DE*	= 2,873.37	D. M. D. *DE* =	2,873.37
		Dep. *DE* =	103.55
Total Dep. *E*	= 1,488.46	Sum =	2,976.92
Total Dep. *F*	= 1,197.83	Dep. *EF* =	−290.63
D. M. D. *EF*	= 2,686.29	D. M. D. *EF* =	2,686.29
		Dep. *EF* =	−290.63
Total Dep. *F*	= 1,197.83	Diff. =	2,395.66
Total Dep. *G*	= 974.43	Dep. *FG* =	−223.40
D. M. D. *FG*	= 2,172.26	D. M. D. *FG* =	2,172.26
		Dep. *FG* =	−223.40
Total Dep. *G*	= 974.43	Diff. =	1,948.86
Total Dep. *A*	= 0	Dep. *GA* =	−974.43
D. M. D. *GA*	= 974.43	D. M. D. *GA* =	974.43 = −Dep. *GA*

the first course of the traverse is equal to the departure of the course, and, as a check on the work, the double meridian distance of the last course should be equal to the departure of the course with its sign changed. The corrected departures for the survey plotted in Fig. 18 are given in Table II. The double meridian distance of the course AB is equal to its departure, or 101.47. The double meridian distance of BC is equal to the double meridian distance of AB, which is 101.47, plus the departure of AB, which is 101.47, plus the departure of BC, which is 847.17; the result is $101.47+101.47+847.17 = 1{,}050.11$, as obtained by taking the sum of the total departures of B and C.

The calculations for determining the double meridian distances of all the courses by both methods are conveniently arranged in the accompanying tabulation. Since the departures of EF, FG, and GA are westings, they are subtracted in applying the rule. The computed value of the double meridian distance of the last course GA is 974.43, and the departure of the course is -974.43. The computations are, therefore, correct because these values are numerically equal and have different signs. Since the meridian is taken through A, which is the most westerly corner, all the double meridian distances are positive in this case. However, if the meridian had been taken through any other corner, some of the values would have been negative.

COMPUTATION OF AREA

41. The area of any polygon can be computed by dividing the given figure into triangles, rectangles, and trapezoids, finding the area of each portion, and taking the sum of these partial areas. Sometimes, it is more convenient to compute the area of a figure that encloses the given polygon and then deduct from it the areas not included in the given polygon.

For instance, in Fig. 18, the area of the field $ABCDEFGA$ may be taken as equal to the area $C'CDEFF'C'$ minus the area $C'CBAC'$ minus the area $AGFF'A$. But area $C'CDEFF'C'$ is equal to $C'CDD'+D'DEE'+E'EFF'$; area $C'CBAC'$ is equal to $C'CBB'+B'BA$; and area $AGFF'A$ is equal to

$G'GFF' - G'GA$. Hence, area $ABCDEFGA$ is equal to $C'CDD' + D'DEE' + E'EFF' - C'CBB' - B'BA - G'GFF' + G'GA$.

The area of a triangle is equal to one-half the product of the base and the altitude, and the area of a trapezoid is equal to one-half the product of the altitude and the sum of the bases: thus, area $B'BA$ is equal to $\frac{1}{2}\ AB' \times B'B$, and area $C'CBB'$ is equal to $\frac{1}{2}\ B'C' \times (B'B + C'C)$. But AB' and BB' are, respectively, the latitude and double meridian distance of AB; $B'C'$ is the latitude of BC, and $B'B + C'C$ is its double meridian distance. Hence, area $B'BA$ is equal to one-half the product of the latitude and the double meridian distance of AB, and area $B'BCC'$ is equal to one-half the product of the latitude and the double meridian distance of BC. Similarly, the other areas are equal to one-half the products of the latitudes and the double meridian distances of the other courses. If the latitude and the double meridian distance are multiplied, the product is the *double area*, since the area itself is equal to one-half the product. It is convenient to consider double areas and then, after they have been combined, to divide the result by 2.

In the preceding expression for area $ABCDEFGA$, it will be noticed that areas $C'CDD'$, $D'DEE'$, $E'EFF'$, and $G'GA$ are to be added, and the courses CD, DE, EF, and GA, which form the inclined sides of the respective areas, all have south latitudes. On the other hand, areas $C'CBB'$, $B'BA$, and $G'GFF'$ are to be subtracted and the courses BC, AB, and FG have north latitudes. Therefore, if all the double meridian distances are positive, the area may be computed by the following rule:

Rule.—*Multiply the double meridian distance of each course by the latitude of the course. Take the sum of the products for the courses having north latitudes and the sum of the products for the courses having south latitudes. Subtract the smaller result from the larger and divide the remainder by 2. The area is taken as positive in every case.*

The calculations for the area of the field in Fig. 18 are shown in Table V. North latitudes and east departures are indicated +, and south latitudes and west departures are indicated −.

TABLE V
CALCULATION OF AREA

Course	Latitude	Departure	D. M. D.	Double Area	
				North	South
AB	+651.42	+101.47	101.47	66,100	
BC	+296.54	+847.17	1,050.11	311,400	
CD	−403.33	+436.27	2,333.55		941,191
DE	−619.96	+103.55	2,873.37		1,781,374
EF	−256.34	−290.63	2,686.29		688,604
FG	+499.76	−223.40	2,172.26	1,085,608	
GA	−168.09	−974.43	974.43		163,792
				1,463,108	3,574,961
					1,463,108
					2)2,111,853
					1,055,926 sq. ft.
					=24.241 acres

EXAMPLES FOR PRACTICE

1. Verify the double meridian distances of the courses for the following notes:

Course	Latitude	Departure	D. M. D.
AB	+521.55	+368.79	368.79
BC	−467.51	+819.20	1,556.78
CD	−712.99	+100.57	2,476.55
DE	+112.01	−596.25	1,980.87
EF	− 97.30	−612.49	772.13
FA	+644.24	− 79.82	79.82

2. Calculate the area of the field in example 1.

Ans. 24.140 acres

CIRCULAR AND PARABOLIC CURVES

Serial 2912A-2 (PART 1) Edition 1

SIMPLE CURVES

PRELIMINARY EXPLANATIONS

TYPES OF CURVES

1. Horizontal Curves.—The ideal location for a railroad or a highway is along a straight and level line from end to end. However, it is seldom practical or economical to achieve this ideal condition because hills and valleys are encountered along the route and it may be necessary either to cross them or to go around them; sometimes it is found most economical to follow the bank of a river. Since automobiles on the highway, or locomotives and cars on the railroad, cannot abruptly change their direction of travel when moving at high speed, it is impractical to construct a road consisting of a series of straight lines with sharp changes of direction at the points of intersection. In order that the road may follow an economical route without sudden changes in direction, the straight portions are connected by curves. The straight portions of the route are commonly called *tangents*, because they are tangent to the connecting curves. The horizontal curves used in railroad and highway work may be divided into the following four general classes: simple curves, compound curves, reverse curves, and easement curves.

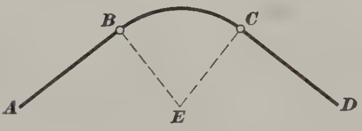

FIG. 1

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2. When two tangents are connected by a single circular arc, the curve is called a *simple curve*. Thus, in Fig. 1, the tangents AB and CD are joined by the simple curve BC whose center is at E and whose radius is EB or EC.

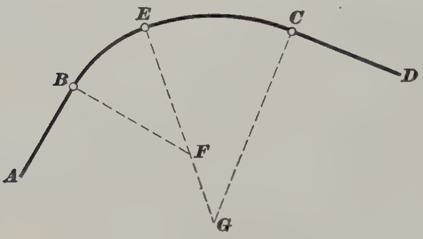

FIG. 2

In some cases, particularly in rough hilly country, the cost of constructing a railroad or highway can be reduced considerably by connecting two tangents by means of a *compound curve*, which is a continuous curve composed of two or more circular arcs curving in the same direction but with different radii. In Fig. 2, the tangents AB and CD are joined by the compound curve BEC, which consists of the two circular arcs BE and EC. The center of the arc BE is at F and its radius is FB or FE; also, the center of the arc EC is at G and its radius is GE or GC.

Occasionally it is necessary to use a *reverse curve*, which consists of two circular arcs curving in opposite directions, as shown in Fig. 3. Here, the tangents AB and CD are connected by the reverse curve BEC, which is composed of the circular arc BE, whose center is at F, and the circular arc EC, whose center is at G. The radii FE and GE of the two arcs may be either equal or unequal. Reverse curves should never be used on trunk highways or railroad main lines, and if possible should be avoided even on roads of less importance.

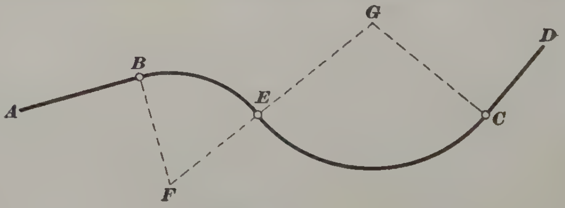

FIG. 3

The abrupt change in direction from a tangent to a sharp circular curve or from a curve to a tangent is not considered desirable in modern railroad and highway construction. Hence, *easement curves*, or *spirals*, along which the curvature changes gradually, are usually introduced at the ends of sharp curves. Such easement curves are also inserted between the parts of a compound curve where one part is much sharper than the other, and between the two curves of a reverse curve. Easement curves are treated in another lesson.

3. Vertical Curves.—The vertical profile of a road usually includes a number of straight lines with different slopes. In order to avoid a sudden change in inclination, it is generally necessary to insert a vertical curve between two adjacent slopes. Such curves are commonly parabolic, because the calculations for establishing the elevations of points along a parabolic vertical curve are especially simple; also, for an ordinary vertical curve, which is extremely flat, a parabola approaches a circular curve in form, and theoretically provides a better transition.

RADIUS AND DEGREE OF CIRCULAR CURVE

4. Designation of Curve by Radius.—The form of a simple circular curve, or the sharpness with which it turns, may be designated by its radius. As the radius is decreased, the curvature becomes sharper. If the radii of highway and railroad curves were relatively short distances, each curve could be conveniently laid out by locating its center point and swinging the tape about the center. However, the curves used in practice are very flat and, consequently, the radii are so large that it is impractical to attempt to lay out a curve in the field by this simple method. Nevertheless, the value of the radius is used for computing other values needed in laying out the curve.

5. Degree of Curve Based on 100-Foot Chord.—In railroad or highway work, the usual term for stating the sharpness of a curve is the degree of curve. The American Railway Engineering Association, railroad engineers in general, and some highway engineers consider the degree of curve as the central angle subtended by a *chord* of 100 feet, that is, the angle formed at the center of the curve by two radii passing through the extremities of a 100-foot chord.

In Fig. 4, the tangents *AB* and *CE* are connected by the simple curve *FG*, and it is assumed that each of the chords *HJ*, *JK*, and *KL* is 100 feet long. The degree of curve is the angle formed at the center *O* by the radii to the ends of any of the 100-foot chords. Thus, the angle *D*, formed by the radii *HO* and *JO*, is the degree of curve. If the angle *HOJ* is 4°, the curve is called a 4-degree curve, or 4° curve; and, if the angle is 1° 30′, the curve is a 1° 30′ curve.

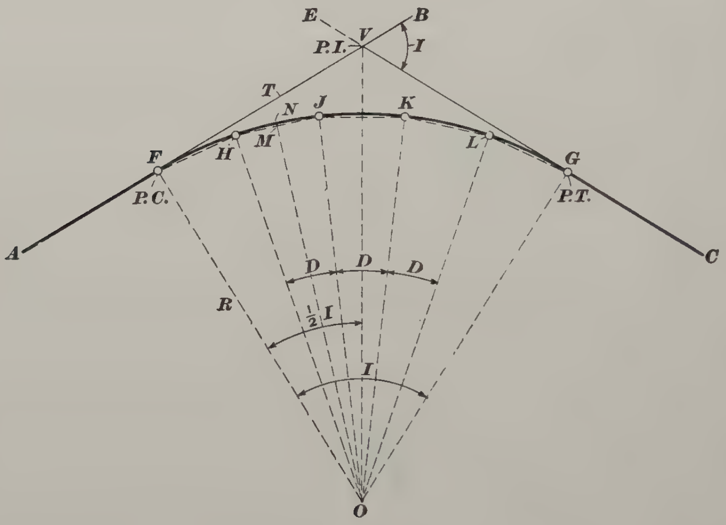

FIG. 4

The relation between the radius of a curve and the degree of curve may be derived as follows: If a line is drawn from the center of the circle to the mid-point *M* of the 100-foot chord *HJ*, the line *OM* is perpendicular to the chord and the angle *HOM* is $\frac{1}{2}HOJ$ or $\frac{1}{2}D$. Hence, in the right triangle *HOM*, $\sin HOM = \frac{HM}{OH}$; or, since $HM = 50$ feet,

$$\sin \tfrac{1}{2}D = \frac{50}{R} \qquad (1)$$

in which D = degree of curve;

R = radius of curve, in feet.

If the degree of curve is known, the radius may be determined by rewriting formula 1, which becomes

$$R=\frac{50}{\sin\frac{1}{2}D} \qquad (2)$$

For small angles the values of $\frac{1}{2}D$, in radians, and $\sin\frac{1}{2}D$ are very nearly equal, or $\sin\frac{1}{2}D$ is approximately equal to $\frac{\frac{1}{2}D}{57.3}$ or $0.00872D$. When this value is substituted in formula 2, the result is $R=50\div 0.00872D$, from which

$$R=\frac{5{,}730}{D} \qquad (3) \textbf{ Approx.}$$

The values of R computed by this formula can be advantageously employed in certain approximate calculations. However, in making computations to establish data for laying out circular curves, it is necessary to use the precise value of R, as found by formula 2.

In Table I, at the end of this lesson, are given the values of the radii for various degrees of curve up to 20°, the degree of curve being taken as the central angle subtended by a chord of 100 feet and the radius being computed by applying formula 2 and using seven-place logarithmic tables. In the case of a curve whose degree is listed in the table, the corresponding radius may be found directly; the radius for any intermediate degree that is not listed can be found by interpolation.

EXAMPLE 1.—The degree of a simple circular railroad curve is 4° 30′. Compute the radius (*a*) accurately and (*b*) approximately.

SOLUTION.—(*a*) In this case, formula 2 is applied and $\frac{1}{2}D=\frac{4^\circ\,30'}{2}=2^\circ\,15'$. If the natural sine of 2° 15′ is taken to only five decimal places,

$$R=\frac{50}{\sin\frac{1}{2}D}=\frac{50}{\sin 2^\circ\,15'}=1{,}273.56 \text{ ft.} \quad \text{Ans.}$$

From Table I, R is found to be 1,273.57 ft.

(*b*) By formula 3,

$$R=\frac{5{,}730}{D}=\frac{5{,}730}{4.5}=1{,}273.3 \text{ ft.} \quad \text{Ans.}$$

EXAMPLE 2.—Find the degree of curve corresponding to a radius of 600 feet.

SOLUTION.—By formula 1,

$$\sin \tfrac{1}{2}D = \frac{50}{R} = \frac{50}{600} = 0.08333$$

Thus, $\frac{1}{2}D = 4° \, 47'$ and $D = 9° \, 34'$. Ans.

6. Degree of Curve Based on 100-Foot Arc.—In most highway work, the degree of curve is considered as the central angle subtended by an *arc* of 100 feet, that is, the angle between two radii passing through the ends of a 100-foot arc. Since the degree of curve D is then subtended by an arc of 100 feet, and the total central angle for a circle, or 360°, is subtended by the circumference of the circle, or $2\pi R$, the following exact proportion is obtained: $\frac{D}{360} = \frac{100}{2\pi R}$, from which $R = \frac{100 \times 360}{2\pi D}$, or

$$R = \frac{5{,}729.58}{D} \qquad (1)$$

When the radius of a curve is given, the degree can be computed by the relation

$$D = \frac{5{,}729.58}{R} \qquad (2)$$

It will be seen that the exact formula 1 is very nearly the same as the approximate formula 3 of the preceding article. Although the methods used for computing and laying out curves are quite similar for the two definitions of degree of curve, the various distances used in the work will differ in value for the same degree of curve. For example, if the degree of curve is taken as the angle subtended by a chord of 100 feet, then the radius of a 4° curve is 1,432.69 feet; whereas, if the degree of curve is the angle subtended by an arc of 100 feet, the radius of a 4° curve is 1,432.40 feet.

In Table II, at the end of this lesson, are given the radii for various degrees of curve up to 20°, the degree of curve being taken as the angle subtended by an arc of 100 feet.

EXAMPLE.—The degree of a simple circular highway curve is 8°. Determine the radius.

SOLUTION.—In this case, it is assumed that the degree of curve is based on an arc of 100 ft. Then

$$R = \frac{5{,}729.58}{D} = \frac{5{,}729.58}{8} = 716.20 \text{ ft.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. If the degree of curve is taken as the angle subtended by a 100-foot chord, what is the radius of a 5° railroad curve as determined by the use of a 5-place table of natural sines? Ans. 1,146.26 ft.

2. Compute the radius of a 15° highway curve, assuming that the degree of curve is based on a 100-foot arc. Ans. 381.97 ft.

3. The radius of a curve is 800 feet. What is the degree of curve, based on a 100-foot chord? Ans. 7° 10′

LOCATING ENDS OF CURVE

7. Required Data for Laying Out Curve.—Before a curve can be laid out on the ground, it is necessary to know the positions of the two adjacent straight portions, or tangents, that are to be connected by the curve and also the degree of the curve. The general procedure is first to locate the tangents as a continuous zig-zag line, and then to select the degree of curve at each change in direction so as to suit the topographic conditions and to satisfy any other requirements that may be important in the particular case. From the degree of curve and the angle between the tangents, may be determined the values that are required in locating first the ends of the curve and then the various intermediate points on the curve. In this lesson, distances are computed to hundredths of a foot, but such accuracy is not always necessary in practice. When the distances are to be laid off in the field only to tenths of a foot, the values should also be computed to tenths.

8. Intersection of Tangents.—The point at which two tangents of a route line intersect is known as the *vertex*, *point of intersection*, or *P. I.*; on drawings, this point is commonly designated by the letter V. In Fig. 4, the two tangents AB and CE are shown intersecting at the vertex V. The external angle formed by the intersecting tangents, as angle BVC, is called the *intersection angle*, or the *angle of intersection*, and is commonly denoted by the letter I. However, some engineers denote the intersection angle by the Greek letter Δ (*delta*).

Since the tangent AB is perpendicular to the radius OF and the tangent CE is perpendicular to the radius OG, the intersection angle BVC between these tangents is equal to the total central angle FOG between the radii passing through the extremities of the curve. In general, *the intersection angle between the tangents through the extremities of any curve is equal to the total central angle of the curve.*

9. Tangent Distance.—The point where the curve begins, or where the first tangent meets the curve, is called the *point of curve* or simply the *P. C.;* the point where the curve ends, or meets the second tangent, is called the *point of tangency* or the *P. T.* From geometry, it is known that the lengths of two tangents to a circle from any point outside the circle are equal; hence, the P. C. and P. T. are equidistant from the P. I. The distance from the P. C. or the P. T. to the P. I. is called the tangent distance, and is usually designated by the letter T. In Fig. 4, the P. C. is at F and the P. T. is at G; and the tangent distance is FV or GV.

The tangent distance bears a direct relation to the radius and the central angle of the curve. In Fig. 4, the line OV from the center of the curve to the vertex bisects the angle FOG and is the hypotenuse of right triangle FVO. Thus, $\tan FOV = \tan \frac{1}{2}FOG = \frac{FV}{FO}$, and $FV = FO \tan \frac{1}{2}FOG$. Since FV is the tangent distance T, FO is the radius R, and FOG is the total central angle, or the angle I,

$$T = R \tan \tfrac{1}{2}I$$

This formula applies to all curves, whether the degree of curve is based on a 100-foot chord or a 100-foot arc.

EXAMPLE.—Two tangents intersecting at an angle of 32° 42′ are to be joined by a 7° 30′ curve. Compute the tangent distance (*a*) for a curve whose degree is measured by a 100-foot chord, and (*b*) for a curve whose degree is determined by a 100-foot arc.

SOLUTION.—(*a*) From Table I, the radius of a 7° 30′ curve is 764.49 ft. Also, in this case, $\frac{1}{2}I = 16° 21'$. Hence,

$$T = R \tan \tfrac{1}{2}I = 764.49 \tan 16° 21' = 224.28 \text{ ft.} \quad \text{Ans.}$$

(*b*) From Table II, $R = 763.94$ ft. For this value,

$$T = 763.94 \tan 16° 21' = 224.12 \text{ ft.} \quad \text{Ans.}$$

10. Length of Curve.—The method of measuring the length of a curve depends on whether the basis for the degree of curve is a 100-foot chord or a 100-foot arc. Where the degree of curve is based on a chord of 100 feet, the length of the curve is the distance from the P. C. to the P. T. measured along a series of successive chords connecting points on the curve. The first chord is from the P. C. to the first 100-foot station on the curve; then follow a number of chords, each 100 feet long, between the first and last 100-foot stations on the curve; the final chord is from the last 100-foot station on the curve to the P. T. Thus, in Fig. 4, the length of the curve *FG* would be measured along the chords *FH*, *HJ*, *JK*, *KL*, and *LG*. The end chords *FH* and *LG* are usually less than 100 feet long, but each of the intermediate chords *HJ*, *JK*, and *KL* must be exactly 100 feet in length.

The central angle subtended by a 100-foot chord is, by definition, equal to the degree of curve *D*, and therefore the angles *HOJ*, *JOK*, and *KOL* are each equal to *D*. In the case of a chord less than 100 feet long, it may be assumed (for the curves ordinarily encountered in railroad practice) that the central angle subtended by the chord is in the same ratio to *D* as the difference between the station numbers of the extremities of the chord is to 100 feet. For example, if the P. C. in Fig. 4 is at Sta. 9+11, the point *H* is at Sta. 10+00, and the degree of curve is 4°, the distance between *F* and *H* is taken as $1{,}000-911=89$ feet, and the central angle *FOH* is found from the relation $FOH : 4° = 89 : 100$. Hence, $FOH = \frac{89}{100} \times 4 = 3.56°$. Similarly, if the P. T. is at Sta. 13+96 and the point *L* is at Sta. 13+00, the distance *LG* is 96 feet and the central angle *LOG* is $0.96 \times 4 = 3.84°$. It therefore follows that, for each chord considered in measuring the length of the curve, the central angle is proportional to the difference between the station numbers of the extremities of the chord. Conversely, the difference between the station numbers at any two points on the curve is proportional to the central angle subtended by the chord joining those points.

Since the total central angle *FOG* is equal to the sum of the angles *FOH*, *HOJ*, *JOK*, *KOL*, and *LOG* that are subtended by the various chords along which the total length of the curve is measured, this total central angle is equal to the product of the degree of curve and the difference, in stations of 100 feet, between the station numbers of the P. C. and P. T. Therefore, the length of the curve in stations can be obtained by dividing the total central, or intersection, angle I by the degree of curve D. Also, the length of curve in feet may be found by the formula

$$L = 100\frac{I}{D}$$

in which L=length of curve, in feet;
I=total central, or intersection, angle;
D=degree of curve.

In applying this formula, it is convenient to express the angles I and D either in minutes or in degrees and a decimal part of a degree rather than in degrees and minutes. Although the formula is approximate, the error involved is negligible for all curves flatter than about 8 degrees, and the formula is therefore much used in practice.

When the degree of curve is considered to be the angle subtended by an arc of 100 feet, the length of the curve is taken as the actual distance along the circular arc. Obviously, the distance between any two points on such a curve is proportional to the central angle subtended by the arc joining the points. Therefore, the preceding formula may be applied for finding the length of any simple curve, whether the degree of curve is based on a chord or an arc of 100 feet.

EXAMPLE.—Determine the length of the curve considered in the example of Art. 9.

SOLUTION.—Here, $I = 32° 42' = 32.7°$ and $D = 7° 30' = 7.5°$. Whether the degree of curve is based on a 100-ft. chord or a 100-ft. arc, the length of the curve is found by the formula

$$L = 100\frac{I}{D} = 100 \times \frac{32.7}{7.5} = 436 \text{ ft.} \quad \text{Ans.}$$

11. Stationing of P. I., P. C., and P. T.—A curve can be laid out without locating the P. I. on the ground, but it is advisable, wherever possible, to determine the station of the P. I. by continuing the straight-line stationing along the first tangent, just as if there were no curve. As a matter of fact, in planning the route, the P. I. is generally established on a map and a suitable curve is then selected, and the station of the P. C. is computed by subtracting from the station of the P. I. the tangent distance for the adopted curve. In any case, the station of the P. T. is found by adding the length of curve to the station of the P. C. The station of the P. T. cannot be determined by adding the tangent distance to the station of the P. I., because the distance from the P. C. to the P. T. along the curve is considerably less than the sum of the distances from the P. C. to the P. I. and from the P. I. to the P. T.

EXAMPLE 1.—If the station of the P. I. for the curve in the examples of Arts. 9 and 10 is 9+31, and the degree of curve is based on a 100-foot chord, determine the station (*a*) of the P. C. and (*b*) of the P. T.

SOLUTION.—(*a*) As calculated in Art. 9, the tangent distance for the curve is 224.28 ft. The distance from Sta. 0+00 to the P. C. is, therefore, $931-224.28=706.72$ ft., and the station of the P. C. is 7+06.72. Ans.

(*b*) The length of the curve, as determined in Art. 10, is 436 ft., and $706.72+436=1{,}142.72$ ft. Consequently, the station of the P. T. is 11+42.72. Ans.

EXAMPLE 2.—Solve example 1 for the case in which the degree of curve is based on a 100-foot arc.

SOLUTION.—(*a*) Here the tangent distance, as found in Art. 9, is 224.12 ft. Since $931-224.12=706.88$ ft., the station of the P. C. is, in this case, 7+06.88. Ans.

(*b*) As determined in Art. 10, the length of this curve is 436 ft. Since $706.88+436=1{,}142.88$ ft., the station of the P. T. is 11+42.88. Ans.

12. Setting P. C. and P. T.—Usually, the first step in laying out a railroad or highway curve on the ground is to set stakes at the P. C. and the P. T. Whenever possible, the P. C. and P. T. are located by measuring the tangent distances along each tangent from the P. I. The position of the P. T. is then established more accurately than if that point were located by running in the curve from the P. C. Stakes at the P. C. and P. T. are driven flush with the ground and the points are accurately marked by tacks on the heads of the stakes. The positions of the P. C. and P. T. are indicated by guard stakes driven about 15 inches off line and left protruding. On these guard stakes are marked the letters P. C. or P. T., as the case may require; and beneath those letters is marked the station number.

13. Summary.—When the intersection angle between two tangents and the degree of the curve that is to connect the tangents are known, the procedure in setting the P. C. and the P. T. on the ground is the same whether the degree of curve is taken as the angle subtended by a 100-foot chord or the angle subtended by a 100-foot arc. In either case, the first step is to determine the radius corresponding to the degree of curve and to compute the tangent distance. Then the tangent distance is laid off from the P. I. along each tangent in turn, and the P. C. and P. T. are located at the ends of these measurements. If the degree of curve is determined by a 100-foot chord, the radius either is taken from a table like Table I or is computed by formula 2, Art. 5; and the tangent distance is found by the formula of Art. 9. For a curve whose degree is determined by a 100-foot arc, the radius either is taken from a table like Table II or is calculated by formula 1, Art. 6; and the tangent distance is also computed by the formula of Art. 9.

For either definition of degree of curve, the length of curve is found by the formula of Art. 10, the station number of the P. C. is found by subtracting the tangent distance from the station number of the P. I., and the station number of the P. T. is found by adding the length of curve, in feet, to the station number of the P. C.

EXAMPLES FOR PRACTICE

1. A 6° 30′ railroad curve, whose degree is determined by a 100-foot chord, connects two tangents that intersect at an angle of 37° 18′. Calculate (*a*) the tangent distance, taking the radius from Table I, and (*b*) the length of curve.

Ans. { (*a*) 297.67 ft.
(*b*) 573.85 ft.

2. If the station number of the P. I. for the curve in the preceding example is 12+13, what is the station number (*a*) of the P. C. and (*b*) of the P. T.?

Ans. { (*a*) 9+15.33
(*b*) 14+89.18

3. Two tangents that intersect at an angle of 24° 48′ are joined by a 4° 15′ curve whose degree is based on a 100-foot arc. Find (*a*) the tangent distance and (*b*) the length of curve.

Ans. { (*a*) 296.40 ft.
(*b*) 583.53 ft.

LOCATING INTERMEDIATE POINTS ON CURVE

MEASUREMENT OF DISTANCES

14. Station Numbers of Intermediate Points.—On curves, as well as on tangents, a stake is generally set at each full station of 100 feet, such as 16+00, 17+00, 18+00, and so on. On sharp curves, it is common practice to set stakes also at 50-foot or 25-foot stations, that is, at stations such as 16+50, 17+25, 17+50, or 17+75 whose numbers are multiples of 50 or 25 feet.

If an obstruction on the curve prevents the setting of a stake at a 100-foot station, a stake is set instead on the curve at a point that is as near as possible to the inaccessible 100-foot station, and the point is denoted by the proper plus stationing. Similarly, where stakes are to be set at intervals of 50 or 25 feet, and for some reason a point at the desired plus station cannot be established, a different point that is on the curve and close to the inaccessible point may be located instead.

There is no general rule for determining what interval should be adopted between successive points on a curve, but a common practice is to use 100-foot intervals for any curve whose degree of curve is less than 8°; 50-foot intervals for curves from 8° to 16°; and 25-foot intervals for curves sharper than 16°.

15. Measurements on Curve Whose Degree is Based on 100-Foot Arc.—As previously explained, the length of a curve whose degree is based on a 100-foot arc is taken as the distance along the arc from the P. C. to the P. T. Also, the distance between any two intermediate points on the curve is understood to be the distance along the arc. However, in laying out the curve on the ground, it is customary to measure the distances along the chords that join the points on the curve rather than along the arcs. Therefore, the length of chord corresponding to the desired length of arc must be determined in each case.

16. Chord Length for 100-Foot Arc.—Where the distance between successive points on a curve is 100 feet and this distance is understood to be along the arc, the corresponding length of chord may be determined by means of a formula that is based on the following considerations: In general, a chord is shorter than the arc that it subtends. For instance, the length of the chord HJ, Fig. 4, is somewhat less than the length of the arc HJ. Also, if a line is drawn from the center of a circle so as to bisect a chord and is continued until it intersects the arc, the line will likewise bisect the arc. Thus, if the line OM that bisects the chord HJ is continued to intersect the curve at N, the point N will bisect the arc HJ and the angle HON or HOM will be $\frac{1}{2}HOJ$. If the length of the arc HJ is 100 feet, the angle HOJ will be equal to the degree of curve D and, consequently, the angle HOM will be $\frac{1}{2}D$. Since the line OM bisects the chord HJ, it is also perpendicular to that chord. In the right triangle HOM, $\sin HOM = \frac{HM}{HO} = \frac{HM}{R}$. If the length of the chord HJ is denoted by C, HM is $\frac{1}{2}C$, and the preceding equation becomes $\sin \frac{1}{2}D = \frac{\frac{1}{2}C}{R}$. Hence,

$$C = 2R \sin \tfrac{1}{2}D$$

in which C = length, in feet, of chord for 100-foot arc;
R = radius of curve, in feet;
D = degree of curve.

In Table II at the end of this lesson are given the chord lengths for 100-foot arcs for various degrees of curve up to 20°.

EXAMPLE.—Calculate the chord length for a 100-foot arc on an 8° curve.

SOLUTION.—From Table II, $R = 716.2$ ft. Then

$$C = 2R \sin \tfrac{1}{2}D = 2 \times 716.2 \times \sin 4° = 99.92 \text{ ft.} \quad \text{Ans.}$$

This value can also be taken directly from Table II.

17. Chord Length for Arc Less Than 100 Feet Long.—It is seldom that the P. C. of a curve comes at a full station. When, as is usually the case, the P. C. is located at a plus station, the first full station on the curve will be less than 100 feet from the P. C. Also, when the P. T. is at a plus station, the distance from the last full station on the curve to the P. T.

will be less than 100 feet. For example, if the P. C. of the curve shown in Fig. 4 is at Sta. 9+11, and the point H is at Sta. 10+00, the arc from F to H is 89 feet long; and, if the P. T. is at Sta. 13+96 and the point L is at Sta. 13+00, the arc from L to G is 96 feet long. Arcs less than 100 feet in length are also used wherever stakes are set at intervals of 50 or 25 feet, or where they are set at plus stations instead of full stations on account of obstructions. An arc is often called a subarc if its length differs from the normal interval between the intermediate points that are located on the curve in the field.

A formula for determining the length of chord for any length of arc can be derived as follows: If the length of the arc HJ in Fig. 4 is assumed to be any distance a, and not 100 feet as in the preceding article, the central angle HOJ will be equal to $\frac{a}{100}\times D$, or $\frac{aD}{100}$, because the central angle is proportional to the length of arc, and the central angle for a 100-foot arc is the degree of curve D. Then, if the length of the chord HJ is denoted by c, HM is $\frac{1}{2}c$, and it follows from the relations in the triangle HOM that $\sin \frac{1}{2}\times\frac{aD}{100}=\frac{\frac{1}{2}c}{R}$. Hence,

$$c=2R\sin\frac{aD}{200}$$

in which c=length of chord for any given arc, in feet;

R=radius of curve, in feet;

a=length of arc, in feet;

D=degree of curve.

In Table II are given the chord lengths for 25- and 50-foot arcs for various degrees of curve up to 20°.

EXAMPLE.—In an 18° curve, whose degree is based on a 100-foot arc, what is the length of the chord subtended by an arc 25 feet long?

SOLUTION.—According to formula 1, Art. 6, $R=\frac{5729.58}{D}=\frac{5729.58}{18}=318.31$ ft. Also, since $a=25$ ft., $\frac{aD}{200}=\frac{25\times18}{200}=2.25°$ or $2°\,15'$. Hence, the required length of chord is

$$c=2R\sin\frac{aD}{200}=2\times318.31\times\sin 2°\,15'=24.99 \text{ ft.}\quad\text{Ans.}$$

This value can also be taken directly from Table II.

18. Measurements on Curve Whose Degree is Based on 100-Foot Chord.—Where the degree of curve is understood to mean the angle subtended by a 100-foot chord, the length of the chord between two successive full stations on the curve is 100 feet. However, where the P. C. or P. T. is not at a full station, the length of the chord at the end of the curve is less than 100 feet. Also, in case it is necessary to set stakes at intermediate points on the curve that are not at full stations, chord lengths of less than 100 feet must be used. Any chord that is less than 100 feet long is known as a *subchord*.

Most curves whose degrees are determined by 100-foot chords are comparatively flat and, therefore, it is generally sufficient to set intermediate stakes only at full stations. On these flat curves, the only subchords are at the P. C. and the P. T. The length of the subchord from the P. C. to the first full station on the curve is assumed to be the difference between 100 and the plus in the station number of the P. C. For example, if the station number of the P. C. is 9+11, the subchord from the P. C. to Sta. 10+00 would be made 89 feet long. Also, the length of the subchord from the last full station on the curve to the P. T. is assumed to be equal to the plus in the station number of the P. T. Thus, if the station of the P. T. is 13+96, the length of the subchord from Sta. 13+00 to the P. T. would be 96 feet.

19. As previously stated, where the degree of curve exceeds 8°, it is customary to set intermediate stakes between the regular 100-foot station points. On a sharp curve whose degree is based on a 100-foot chord, it is necessary to make allowance for the difference between the length of each subchord and the station interval between the ends of the subchord. When points on a curve are to be located at intervals of 50 feet, each 50-foot station must be at the center of the arc between adjacent full stations; and, if stakes are to be set at intervals of 25 feet, the arc between each two stakes must be one-quarter of the arc between two 100-foot stations. For example, if the chord *AB* in Fig. 5 is 100 feet long and stakes are to be set on the curve at 25-foot intervals, the 25-foot station points C, D, and E must be located so as to divide the arc AB into four equal parts, or so that all four chords AC, CD, DE, and EB will be of equal length. Since the sum of the lengths of these four chords is obviously greater than the length of the long chord AB, or 100 feet, it is evident that the length c of each short chord is greater than 25 feet. However, the stations at C, D, and E are numbered as if each chord is exactly 25 feet long. In general, the actual length of any subchord of a curve whose degree is based on a 100-foot chord is greater than the difference between the station numbers at the extremities of the subchord. The actual length of the subchord between any two stations may be found in the following manner:

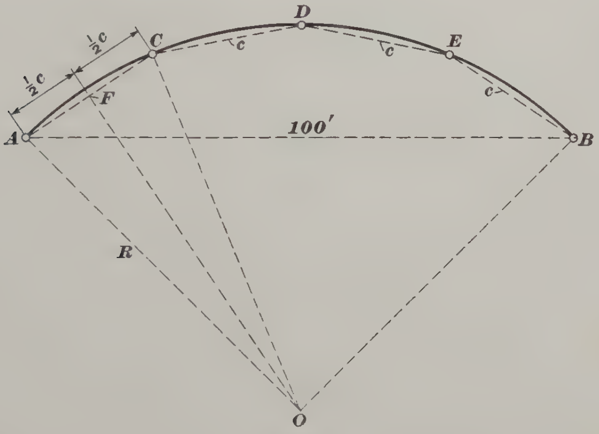

FIG. 5

For all practical purposes it may be assumed that the difference c' between the station numbers at the two extremities of the subchord is in the same ratio to 100 feet, or the difference between two adjacent full stations, as the central angle subtended by the subchord is to the degree of curve D. Thus, for the curve shown in Fig. 5, $\frac{c'}{100}=\frac{AOC}{D}$, and it follows that the central angle AOC subtended by the subchord is equal to $\frac{c'}{100}\times D$. Also, if a perpendicular bisector OF is drawn from O

to the subchord AC, and the right triangle AOF is solved as in Art. 16 or 17, it is found that

$$c=2R\sin\frac{c'D}{200}$$

in which c = actual length of subchord, in feet;
R = radius of curve, in feet;
c' = difference between station numbers at extremities of subchord, in feet;
D = degree of curve.

EXAMPLE 1.—What is the actual length of the subchord between Stas. 18 and 18+50 on a 12° curve whose degree is based on a 100-foot chord?

SOLUTION.—From Table I, it is found that the radius of a 12° curve is 478.34 ft. Also, in this case, $c'=50$ ft., $D=12°$, and $\frac{c'D}{200}=\frac{50\times 12}{200}=3°$. Therefore, the required length of chord is

$$c=2R\sin\frac{c'D}{200}=2\times 478.34\times\sin 3°=50.07\text{ ft.}\quad\text{Ans.}$$

EXAMPLE 2.—If the P. C. of the curve in example 1 is at Sta. 38+77, what is the actual length of the chord from the P. C. to Sta. 39+00?

SOLUTION.—Here, $c'=23$ ft., $D=12°$, $R=478.34$ ft., and $\frac{c'D}{200}=\frac{23\times 12}{200}$ $=1.38°$, or 1° 22.8′. Then, the actual length of the chord is

$$c=2\times 478.34\times\sin 1°\ 22.8'=2\times 478.34\times 0.02408=23.04\text{ ft.}\quad\text{Ans.}$$

EXAMPLES FOR PRACTICE

1. Calculate the chord length for an arc 100 feet long on a 6° 10′ curve whose degree is based on a 100-foot arc. Ans. 99.95 ft.

2. If the P. C. of the curve in the preceding example is at Sta. 7+28, what is the length of the chord from the P. C. to Sta. 8+00? Ans. 71.99 ft.

3. The P. T. of a 15° curve whose degree is based on a 100-foot chord is at Sta. 16+87. Determine the actual length of the chord from Sta. 16+50 to the P. T. Ans. 37.10 ft.

DEFLECTION ANGLES WITH TRANSIT AT P. C.

20. Methods of Locating Points on Simple Curves.—Five commonly used methods of locating points on simple curves are as follows: (1) by deflection angles; (2) by tangent offsets, or offsets from the tangent; (3) by ordinates from the long chord; (4) by deflection distances, or offsets from chords produced; and (5) by middle ordinates.

The method by deflection angles is generally the simplest and most convenient, and is used whenever possible. However, where thick woods make the method by deflection angles difficult, either the method of offsets from the tangent or the method of offsets from the long chord is generally used. Another factor that limits the use of the method of deflection angles is the necessity of employing a transit for locating points on the curve. A transit is not essential for the other methods, as the curve may be located with a tape and plumb-bobs. When the curve is to be laid out in open country without the use of a transit, and the degree of curve is based on a 100-foot chord, the method commonly used for locating the points is either by middle ordinates or by deflection distances, but neither of these two methods is well adapted for laying out a curve whose degree is based on a 100-foot arc.

21. Deflection Angles From Tangent at P. C.—The angle between the tangent at any point on a curve and the chord from that point to any other point on the curve is commonly known as the deflection angle from the tangent to the second point. In other words, the deflection angle is the angle by which the chord is deflected from the tangent. In Fig. 6, the tangents AB and CD, which intersect at V, are connected by a simple curve BC, whose center is at O; the P. C. is at B, the P. T. is at C, and the intermediate 100-foot stations are at E, F, G, and H. The deflection angle from the tangent AV at the P. C. to the point E is the angle VBE between that tangent and the chord BE. Also, the deflection angles from the tangent AV to the other 100-foot stations on the curve and to the P. T. are VBF, VBG, VBH, and VBC.

From geometry, the angle between a tangent to a circle and any chord through the point of tangency is equal to one-half of the central angle subtended by the chord. Hence, the deflection angle from the tangent at the P. C. to any station on the curve, or the angle between that tangent and the chord from the P. C. to the station, is equal to one-half of the central angle subtended by that chord.

22. General Method for Calculating Deflection Angles From Tangent at P. C.—Whether the degree of curve is defined as the central angle subtended by a 100-foot chord or as that subtended by a 100-foot arc, *the central angle subtended by any chord of a simple curve is equal to the product of the degree of curve and the difference, in stations of 100 feet, between the station numbers of the extremities of the chord.* Thus, if D represents the degree of curve and s represents the distance in stations between the extremities of the chord, the central angle subtended by the chord is equal to sD. Since the deflection angle from the tangent at the P. C. to any station on the curve is equal to one-half the central angle subtended by the chord from the P. C. to the point, that deflection angle may be computed by the formula

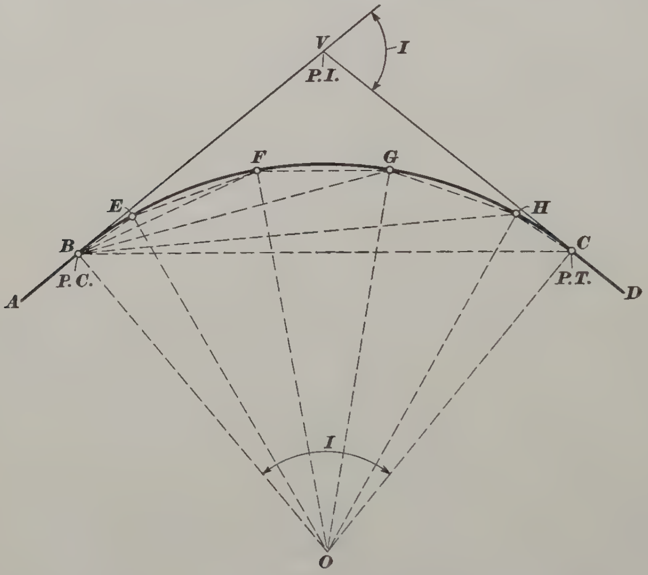

FIG. 6

$$d = \frac{sD}{2}$$

in which d = deflection angle from tangent at P. C. to any point on curve;

s = difference, in stations of 100 feet, between station numbers of P. C. and point on curve;

D = degree of curve.

For given values of s and D, the deflection angle is the same whether the degree of curve is based on a 100-foot chord or a 100-foot arc.

EXAMPLE.—The P. C. of a 6° curve is at Sta. 15+27. Determine the deflection angle from the tangent at the P. C. to each of the following points on the curve: (*a*) Sta. 16+00, (*b*) Sta. 17+00, and (*c*) Sta. 20+98.

SOLUTION.—(*a*) The distance in 100-ft. stations from the P. C. to Sta. 16+00 is $16-15.27=0.73$, and the required deflection angle is

$$d=\frac{sD}{2}=\frac{0.73\times 6}{2}=2.19° \text{ or } 2°\ 11.4'. \quad \text{Ans.}$$

(*b*) The distance from Sta. 15+27 to Sta. 17+00 is $17-15.27=1.73$ sta., and in this case the deflection angle is

$$d=\frac{1.73\times 6}{2}=5.19° \text{ or } 5°\ 11.4'. \quad \text{Ans.}$$

(*c*) Here, $s=20.98-15.27=5.71$ sta., and

$$d=\frac{5.71\times 6}{2}=17.13° \text{ or } 17°\ 7.8'. \quad \text{Ans.}$$

23. The method of computing deflection angles explained in the preceding article is correct for any station on a curve. However, the deflection angle to each station is computed independently, so that there is no convenient check on the calculations as a whole; also, for each point it is necessary to find the difference between the station number of that point and the station number of the P. C., and usually each value of s contains a decimal part. For these reasons, the following method has been developed for computing deflection angles for consecutive points on curves.

In Fig. 6 it is assumed that the P. C. at B and the P. T. at C are at plus stations, and that E, F, G, and H are at the intermediate 100-foot stations. The deflection angle VBE from the tangent BV at the P. C. to the first full station E is determined by the formula of the preceding article. To determine the deflection angle VBF to the second station, it is convenient to take the sum of the angles VBE and EBF. This latter angle is an inscribed angle of the circle; that is, the vertex B is on the circumference of the circle and the sides BE and BF are chords of the circle. Hence, the angle EBF is measured by one-half of the arc EF and is equal to one-half of the central angle EOF. Since E and F are 100-foot stations, the angle EOF is equal to the degree of curve and the angle EBF is one-half of the degree of curve. Consequently, the deflection angle to point F is equal to the sum of the deflection angle to E and one-half of the degree of curve.

Similarly, the deflection angle VBG to the third station G is equal to $VBF+FBG$. Since FBG is an inscribed angle and is measured by one-half of the arc FG, it is equal to one-half of the central angle FOG, or to one-half of the degree of curve. Therefore, the deflection angle to G is equal to the deflection angle to F plus one-half of the degree of curve. Since this procedure may be continued for each full station on the curve, it may be concluded that the following general rule applies to any simple curve:

Rule.—*The deflection angle from the tangent at the P. C. to any 100-foot station on a simple curve is equal to the sum of the deflection angle to the preceding 100-foot station and one-half of the degree of curve.*

The deflection angles to all full stations on the curve can be readily found by applying the preceding rule. Also, a simple and reliable check on these values can be obtained by continuing the computations to determine the deflection angle to the P. T. and then finding that angle by another method. It is obvious from Fig. 6 that the deflection angle VBC to the P. T. is equal to $VBH+HBC$, and HBC is equal to one-half of the central angle HOC. Hence, the deflection angle to the P. T. may be found by obtaining the sum of the deflection angle previously computed for the last 100-foot station on the curve and one-half of the central angle subtended by the chord between that station and the P. T. It is also evident from Fig. 6 that the deflection angle VBC is equal to one-half of the total central angle BOC, or $\frac{1}{2}I$. Therefore, the deflection angle to the P. T. as found by the longer method is compared with $\frac{1}{2}I$ and, if the values agree, the computations for the deflection angles to all intermediate 100-foot stations may be assumed to be correct.

Example.—Determine the deflection angle from the tangent at the P C. to each 100-foot station on the curve in Art. 22 and check the calculations by determining the deflection angle to the P. T., which is at Sta. 22+47. The angle of intersection between the tangents is 43° 12′.

Solution.—As found in Art. 22, the deflection angle to Sta. 16+00 is 2° 11.4′. Since $\frac{1}{2}D=\frac{1}{2}\times 6°=3°$, the deflection angles to the other full stations on the curve are:

To Sta. 17+00, $2° 11.4'+3°=5° 11.4'$
To Sta. 18+00, $5° 11.4'+3°=8° 11.4'$
To Sta. 19+00, $8° 11.4'+3°=11° 11.4'$
To Sta. 20+00, $11° 11.4'+3°=14° 11.4'$
To Sta. 21+00, $14° 11.4'+3°=17° 11.4'$
To Sta. 22+00, $17° 11.4'+3°=20° 11.4'$

The central angle subtended by the chord from Sta. 22 to the P. T. at Sta. 22+47 is

$$sD=0.47\times 6=2.82° \text{ or } 2° 49.2'$$

Hence, the deflection angle between the tangent at the P. C. and the chord from the P. C. to the P. T. is

$$20° 11.4'+\tfrac{1}{2}\times 2° 49.2'=21° 36'$$

Since $\frac{1}{2}I=\frac{1}{2}\times 43° 12'=21° 36'$, the values of the deflection angles previously computed may be assumed to be correct.

24. The principle of the method described in the preceding article can also be applied when stakes are set on the curve at intervals of 50 feet or 25 feet. However, the increment to be added to the deflection angle of the preceding point for each 50-foot interval is $\frac{1}{4}D$, or one-half of the increment for a 100-foot interval; and the increment to be added to the deflection angle for each 25-foot interval is $\frac{1}{8}D$. If, for example, in a 12° curve the deflection angle to Sta. 16+00 is 8° 14′, then $\frac{1}{4}D=3°$ and the deflection angle to Sta. 16+50 is $8° 14'+3° =11° 14'$.

25. In the case of a subchord or subarc at the beginning of a curve, it may be convenient to compute the deflection angle directly in minutes. Then, since a station length contains 100 feet and a degree contains 60 minutes, the required angle may be found by substituting in the formula of Art. 22 the following values: $\frac{d'}{60}$ for d and $\frac{c'}{100}$ for s, where d' is the required deflection angle in minutes and c' is the difference in feet between the station numbers of the extremities of the subchord. Thus,

$\frac{d'}{60}=\frac{\frac{c'}{100}\times D}{2}$, from which

$$d'=0.3\,c'\,D$$

EXAMPLE.—If the P. C. of a 3° curve is at Sta. 37+84, what is the deflection angle, in minutes, to Sta. 38+00?

SOLUTION.—In this case, $c'=16$ ft., and

$d'=0.3\,c'\,D=0.3\times 16\times 3=14.4'$. Ans.

26. Length of Chord at Extremity of Curve.—When the P. C. is not at a full station, as is usually the case, the length of the chord from the P. C. to the next station may be found by applying the formula of Art. 17 or 19. However, when the curve is located by deflection angles, it is generally more convenient to use a formula involving the deflection angle to the first station, because that angle will be known. Such a formula may be derived in the following manner: In Fig. 7, the tangent AV at the P. C. and the tangent BV at the P. T. intersect at V, the center of the curve is at O, the first 100-foot station is at C, and the last 100-foot station is at D. If the chord AC and its perpendicular bisector OE are drawn, then the angle AOE will be equal to the deflection angle VAC from the tangent at the P. C. to the point C, because the sides of one angle are perpendicular to the sides of the other angle. If c denotes the length of the chord AC, R the radius of the curve, and d the deflection angle VAC, it follows from the relation $AE=OA\sin AOE$ that $\frac{1}{2}c=R\sin d$, or

$$c=2R\sin d$$

The perpendicular bisector OF of the chord BD from the P. T. to the last 100-foot station on the curve also bisects the central angle DOB subtended by that chord. Then, $BF=OB\sin BOF=R\sin\frac{1}{2}DOB$, or *the length of the subchord from* *the last 100-foot station on the curve to the P. T. equals twice the product of the radius of the curve and the sine of one-half the central angle subtended by that subchord.* When the values of the deflection angles are checked by calculating the deflection angle to the P. T., it is necessary to determine one-half the central angle subtended by the subchord from the last full station to the P. T. Therefore, in computing the length of that subchord, one-half the central angle subtended by the subchord is known and need not be recalculated.

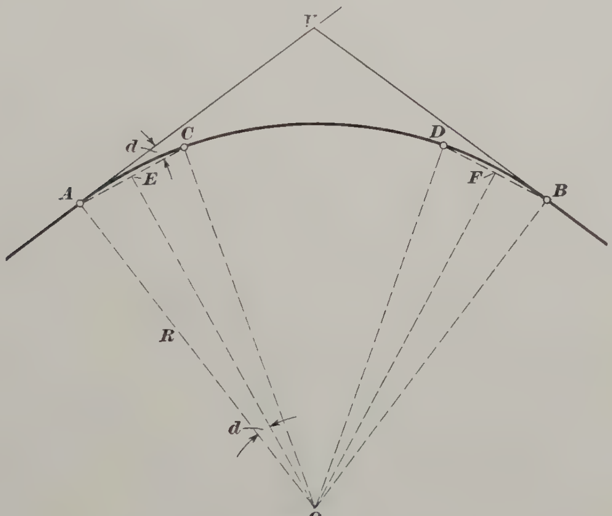

FIG. 7

27. Setting 100-Foot Stations by Deflection Angles.—Whenever possible, it is desirable to lay out the entire curve with the transit set at the P. C. Before the field work for locating the intermediate points on the curve is started, the deflection angles from the tangent at the P. C. to the various stations on the curve are computed. Then, the transit is set up at the P. C. and, with the vernier reading zero, the line of sight is brought into the tangent through the P. C. For this purpose, the line of sight should preferably be directed to the P. I.; but, in case that point is not visible from the P. C., any point on the tangent may be used instead. For instance, if it is desired to lay out the curve BC in Fig. 6, the transit is set up at B, the vernier is turned to zero, the telescope is directed to V or to any other point on the tangent BV, and the lower plate of the instrument is clamped.

Whether the degree of curve is based on a 100-foot chord or on a 100-foot arc, the 100-foot stations on the curve can then be located as follows: To set the point E, the telescope is turned—in this case clockwise, because the curve lies to the right of the tangent BV—so that the vernier reading is equal to the computed deflection angle from the tangent BV to point E; and, along the line of sight thus obtained, the required length of the chord BE is measured from the P. C. A stake, which has the station number written on it, is driven part way into the ground at the end of the chord thus measured.

In order to locate the point F, the vernier is set to read the deflection angle VBF, and the line of sight is thereby directed along the chord BF; one end of a tape is then held at E while the tape is stretched taut and swung so that a point at a distance from E equal to the required length of the chord EF is brought exactly into the line of sight. A stake marked with the proper station number is driven at the point thus located. The point G is located in a similar manner by setting the vernier to read the deflection angle VBG and measuring from F a distance equal to the required length of the chord FG so that the end of the measurement is in the line of sight from the transit. Likewise, the point H is established by setting the vernier to read the deflection angle VBH and measuring the chord GH.

Although the P. T. has already been located, it is always advisable to turn off the total deflection angle VBC on the vernier and to measure the final chord HC, in order to see how closely the point thus determined comes to the previously established position of the P. T. If the two points do not coincide exactly, but the error is reasonably small, the position of the P. T. as determined from the P. I. should be used as the final point on the curve, and the positions of the intermediate 100-foot stations as determined from the P. C. need not be changed. In case the error at the P. T. is comparatively large, the curve should be rerun from the P. C., and the

100-foot stations that were at first located inaccurately should be moved. An additional check on the work may be had by setting up at the P. T. and measuring the angle *BCV* between the chord *CB* to the P. C. and the tangent *CV* to the P. I.

EXAMPLE 1.—Two tangents that intersect at an angle of 32° 42′ at Sta. 9+31, as shown in Fig. 8, are to be connected by a 7° 30′ curve whose degree is based on a 100-foot arc. Explain how the curve would be laid out on the ground by deflection angles from the tangent at the P. C.

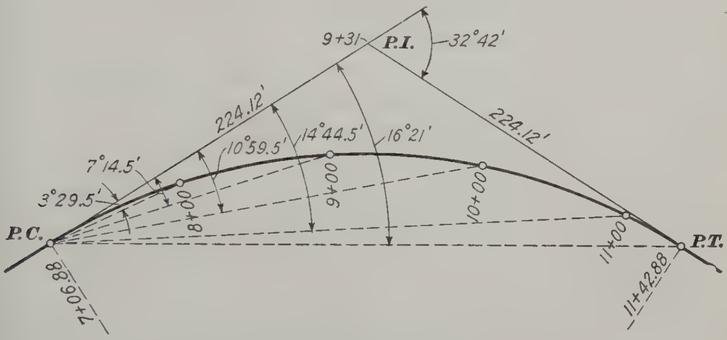

FIG. 8

SOLUTION.—The first step is to determine the station numbers of the P. C. and the P. T., and to establish those points on the ground by measurements from the P. I. In this case, the radius of the curve, which may be taken from Table II or computed by formula 1, Art. 6, is 763.94 ft.; and, by the formula of Art. 9, the tangent distance is

$$T = R \tan \tfrac{1}{2} I = 763.94 \tan 16^\circ\, 21' = 224.12 \text{ ft.}$$

Hence, the P. C. will be at Sta. 7+06.88. Also, by the formula of Art. 10, the length of the curve is

$$L = 100\frac{I}{D} = 100 \times \frac{32.7}{7.5} = 436 \text{ ft.}$$

and the P. T. will be at Sta. 11+42.88. The conditions are represented in Fig. 8, but for convenience the curve is shown much sharper than the true curve.

The deflection angles to the successive 100-ft. stations on the curve are:

Sta. 8, $0.3\,c'D = 0.3 \times 93.12 \times 7.5 = 209.5'$, or $3^\circ\, 29.5'$
Sta. 9, $3^\circ\, 29.5' + 3^\circ\, 45' = 7^\circ\, 14.5'$
Sta. 10, $7^\circ\, 14.5' + 3^\circ\, 45' = 10^\circ\, 59.5'$
Sta. 11, $10^\circ\, 59.5' + 3^\circ\, 45' = 14^\circ\, 44.5'$

The central angle subtended by the subchord from Sta. 11 to the P. T. is $0.4288 \times 7.5 = 3.216^\circ$, or $3^\circ\, 13'$, and $\frac{1}{2} \times 3^\circ\, 13' = 1^\circ\, 36.5'$; then the deflection angle to the P. T. is 14° 44.5′+1° 36.5′=16° 21′. This value is equal to $\frac{1}{2}I$, or $\frac{1}{2}\times 32°\ 42'$, and the preceding calculations may therefore be assumed to be correct.

As this curve is one whose degree is based on a 100-foot arc, it is necessary to determine the chord length from the P. C. to Sta. 8; the chord length for a 100-foot arc; and also, for checking the position of the P. T., the chord length from Sta. 11 to the P. T. By the formula of the preceding article, the length of the chord from the P. C. to Sta. 8 is

$$c=2R\ \sin\ d=2\times 763.94\times \sin\ 3°\ 29.5'=93.06\text{ ft.}$$

Also, by the formula of Art. 16, the length of chord for a 100-foot arc is

$$C=2R\ \sin\ \tfrac{1}{2}D=2\times 763.94\times \sin\ 3°\ 45'=99.93\text{ ft.}$$

To find the length of the subchord from Sta. 11 to the P. T., the principle of the preceding article is applied. As previously determined, the value of one-half of the central angle subtended by the subchord is 1° 36.5′. Hence, the length of the subchord is

$$2\times 763.94\times \sin\ 1°\ 36.5'=42.88\text{ ft.}$$

The intermediate 100-ft. stations on the curve may now be established on the ground in the following manner: The transit is set up at the P. C. and, with the vernier reading zero, the telescope is brought into the tangent at the P. C. by sighting to the P. I. Here, the length of the subchord to Sta. 8 is 93.06 ft., and the deflection angle to Sta. 8 is 3° 29.5′. Hence, to locate Sta. 8, the vernier is turned, in the proper direction, so as to read 3° 29.5′ and a length of 93.06 ft. is laid off along the line of sight. The deflection angle for Sta. 9 is 7° 14.5′ and the length of the chord from Sta. 8 to Sta. 9 is 99.93 ft. Therefore, Sta. 9 is located by setting the vernier at 7° 14.5′, and measuring 99.93 ft. from Sta. 8 so that the end of the chord is on the line of sight from the transit. In a similar manner, Sta. 10 is located by making the vernier reading 10° 59.5′ and measuring 99.93 ft. from Sta. 9; and Sta. 11 is established by setting the vernier at 14° 44.5′ and measuring 99.93 ft. from Sta. 10. If the work is correct, the deflection angle to the P. T. at Sta. 11+42.88 should now be found to be 16° 21′, and the length of the chord from Sta. 11 to the P. T. should be 42.88 ft.

Example 2.—If the tangents in the preceding example are to be connected by a 7° 30′ curve whose degree is based on a 100-foot chord, how would the curve be laid out on the ground by deflection angles?

Solution.—The general method of procedure is exactly the same as that described in the preceding example. For this curve, however, the radius is taken from Table I and is 764.49 ft., and $T=764.49\ \tan\ 16°\ 21'$ $=224.28$ ft. Hence, the P. C. is at Sta. 7+06.72. The length of the curve is 436 ft., as in the preceding example, and the P. T. is then at Sta. 11 +42.72. The conditions are as shown in Fig. 9.

In this case, the deflection angles from the tangent at the P. C. to the various 100-ft. stations on the curve are as follows:

Sta. 8, $0.3\,c'D = 0.3 \times 93.28 \times 7.5 = 209.9'$, or 3° 29.9′
Sta. 9, 3° 29.9′+3° 45′=7° 14.9′
Sta. 10, 7° 14.9′+3° 45′=10° 59.9′
Sta. 11, 10° 59.9′+3° 45′=14° 44.9′

Also, the central angle subtended by the subchord from Sta. 11 to the P. T. is $0.4272 \times 7.5 = 3.204°$ or 3° 12.2′; $\frac{1}{2} \times$ 3° 12.2′=1° 36.1′; and the deflection angle to the P. T. is 14° 44.9′+1° 36.1′=16° 21′, which is equal to $\frac{1}{2}I$.

Since the degree of curve is less than 8°, the length of the chord from the P. C. to Sta. 8 is made 93.28 ft. and the length of the subchord from Sta. 11 to the P. T. should be 42.72 ft.

After the transit has been set up at the P. C. and the line of sight has been brought into the tangent with the vernier reading zero, as described in the preceding example, Sta. 8 is located by turning off the deflection angle of 3° 29.9′ and measuring 93.28 ft. along the line of sight. Then, Sta. 9 is established by setting the vernier to read 7° 14.9′ and measuring 100 ft. rom Sta. 8; Sta. 10, by turning the vernier to 10° 59.9′ and measuring 100 ft. rom Sta. 9; and Sta. 11, by turning the vernier to 14° 44.9′ and measuring 00 ft. from Sta. 10. Finally, as a check on the work, the deflection angle o the P. T. should be 16° 21′, and the chord from Sta. 11 to the P. T. hould be 42.72 ft. long.

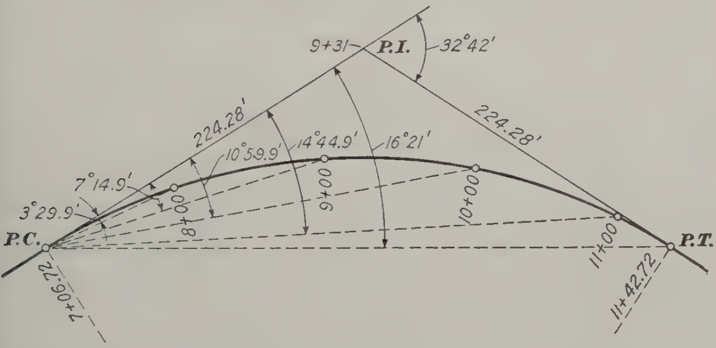

FIG. 9

28. Setting Plus Stations.—The procedure for locating tations on a curve at intervals of 50 or 25 feet is the same s that for locating only 100-foot stations. Before any field ocation is started, the deflection angles to all points that are to e marked on the ground are computed. After the P. C. and . T. have been established, the transit is set up at the P. C. and he first point on the curve is located by turning off the proper deflection angle and measuring the required length of chord along the line of sight of the telescope. The next point is located by turning off the deflection angle for that station and measuring from the previously located station the correct length of chord; at the point where the chord measurement intersects the line of sight of the transit, the new station is established. In each case, the true length of chord, as determined by the formula in Art. 17, 19, or 26, must be used.

Station	Point	Description of Curve	Deflection Angle	Chord	Arc
14				N 14°15'E	N 14°13'E
13					
12					
+42.88	P.T.	I=32°42'	16°21'	42.88	42.88
11		D=7°30'R	14°44.5'	99.93	100.00
10		R=763.94'	10°59.5'	99.93	100.00
9	9 + 31 V	T=224.12'	7°14.5'	99.93	100.00
8		L=436.00'	3°29.5'	93.06	93.12
+06.88	P.C.			N 18°30'W	N 18°29'W
7					
6					
5					

FIG. 10

29. Field Notes.—It is customary to record the field notes for curves in the type of notebook commonly used for ordinary transit surveys. Various methods of arranging the notes are in general use, but the data included are essentially the same in all. In Fig. 10 is shown a typical set of notes for a portion of a route line containing the simple curve considered in example 1, Art. 27. The degree of this curve is based on an arc of 100 feet and the entire curve was laid out from the P. C. Only the left-hand page of the notebook is illustrated. The right-hand page is used for sketches and notations in connection with crossings of streams, fences, etc., and for information concerning topography and other features that may seem important to the engineer.

In the first column of the left-hand page of the notebook are recorded the station numbers, which increase from the bottom of the page upwards. In the second column are the descriptions of the ends of the curve; also, in this column is frequently recorded the station number of the vertex, as determined by continuing the stationing along the first tangent, although this point is not on the curve. The third column contains descriptive data for the curve, which include the intersection angle I; the degree of curve D, followed by the abbreviation R or L to show whether the curve turns to the right or to the left; the radius R; the tangent distance T; and the length of curve L. The fourth column is for the deflection angles to the stations on the curve. In the sample notes the values of R, T, and L, and the stations of the P. C. and P. T. are computed to hundredths of a foot. However, as previously stated, it often is sufficiently accurate to express each of these values only to the nearest tenth.

In the form of notes here illustrated, the last two columns contain, respectively, the chord and arc lengths in feet between the adjacent points on the curve. The bearing of each tangent, both as observed on the compass circle of the transit and as calculated from the bearing of the preceding tangent and the measured intersection angle, must also be included in the notes. As there are only six columns on a page of the notebook, the bearings are often written vertically in the columns headed Chord and Arc, as indicated in Fig. 10. This arrangement is entirely satisfactory, because the lengths of chords and arcs are given only at stations on curves, whereas the bearings of tangents are shown only between curves. However, the values of the chord and arc lengths are sometimes omitted from the notes, and the last two columns are used only for the observed and calculated bearings. Those columns are then given appropriate headings and the values of the bearings are written on horizontal lines just above the station of each P. T.

30. The notes for a curve whose degree is based on a 100-foot chord are similar to those for a curve whose degree is based on a 100-foot arc. In Fig. 11 is shown a typical set of notes for a portion of a railroad line containing a curve whose degree is determined by a 100-foot chord and for which the angles and distances are given or found in example 2, Art. 27. The first four columns in this case contain the same kind of information as do the corresponding columns in Fig. 10. In the last two columns are given the observed and calculated bearings of the tangents. As explained in the preceding article, the last two columns of the notes for a curve whose degree is based on a 100-foot arc may also be similar to those shown in Fig. 11 instead of as shown in Fig. 10. When this method is used the two sets of notes are similar.

Station	Point	Description of Curve	Deflection Angle	Magnetic Bearing	
				Observed	Calculated
14					
13					
12				N 14°15'E	N 14°13'E
+42.72	P.T.	I=32°42'	16°21'		
11		D=7°30'R	14°44.9'		
10		R=764.49'	10°59.9'		
9	9+31 V	T=224.28'	7°14.9'		
8		L=436.00'	3°29.9'		
+06.72	P.C.				
7					
6					
5				N 18°30'W	N 18°29'W

FIG. 11

EXAMPLES FOR PRACTICE

1. The P. C. of a 3° curve, whose degree is based on a chord of 100 feet, is at Sta. 19+39.60 and the P. T. is at Sta. 25+64.60. Calculate the deflection angle from the tangent at the P. C. to each 100-foot station on the curve and to the P. T.

$$\text{Ans.}\begin{cases} 54.4', 2^\circ 24.4', 3^\circ 54.4', 5^\circ 24.4', \\ 6^\circ 54.4', 8^\circ 24.4', 9^\circ 22.5' \end{cases}$$

2. The P. C. of a 14° curve, whose degree is based on an arc of 100 feet, is at Sta. 49+27.36, and the angle of intersection of the tangents is 60° 58′. Calculate the deflection angle from the tangent at the P. C. to each 50-foot station on the curve.

$$\text{Ans.}\begin{cases} 1^\circ 35.1', \quad 5^\circ 05.1', \quad 8^\circ 35.1', \\ 12^\circ 05.1', 15^\circ 35.1', 19^\circ 05.1', \\ 22^\circ 35.1', 26^\circ 05.1', 29^\circ 35.1' \end{cases}$$

DEFLECTION ANGLES WITH TRANSIT AT INTERMEDIATE POINT ON CURVE

31. General Method of Procedure.—It frequently happens, especially in hilly or timbered country, that there are obstructions in the line of vision along a curve so that only a portion of the curve can be run in by deflection angles with the transit at the P. C. Under such conditions, the procedure for running in the curve is as follows: First, the transit is set up at the P. C., and as much of the curve as possible is laid out by turning deflection angles from that point. At the last station that is visible from the P. C., a hub is driven and the point is marked accurately by a tack. The transit is then set up at this point and subsequent points on the curve are located by turning deflection angles with the transit in the new position. Sometimes, it is necessary to set up at more than one intermediate point on the curve.

There are two common methods of procedure in turning off deflection angles when the transit is set at an intermediate point on the curve. In one method, which is known as the *method by deflection angles from an auxiliary tangent*, the intermediate point is used as a second P. C. and the deflection angles are computed with respect to the tangent to the curve at that point. In the other method, called the *method by continuous deflection angles*, all deflection angles are calculated as if the entire curve were laid out with the transit at the P. C. To illustrate the procedure with either method, it is assumed that the tangents *AB* and *CD*, Fig. 12, are to be connected by a simple curve *BC*, that the entire curve is not visible from the P. C. at *B*, and that the transit is set at the intermediate station *E*. Then, if the method by deflection angles from an auxiliary tangent is employed, the next station *F* is located by bringing the line of sight into the chord *EF* with the vernier reading the deflection angle *GEF* that the chord *EF* makes with the tangent *EG* at *E*. In the method by continuous deflection angles, the line of sight is brought into the chord *EF* with the vernier reading the deflection angle *VBF* that the chord *BF* makes with the tangent *BV* at the P. C.

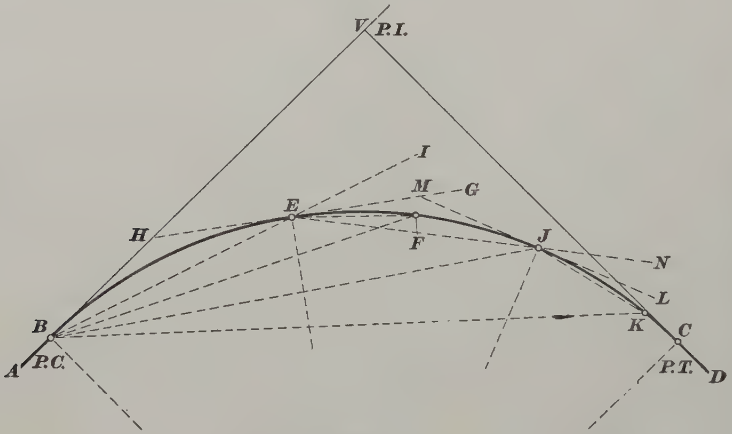

FIG. 12

The station number of the intermediate transit point is not known beforehand in the office. Therefore, when the method by deflection angles from an auxiliary tangent is used, it is necessary to compute in the field the deflection angles to all stations beyond the first intermediate transit point. On the other hand, when the method by continuous deflection angles is employed, computations in the field are not necessary, because the deflection angles for locating all stations on the curve may be previously determined in the office. For that reason, many engineers prefer the method by continuous deflection angles. However, when the other method is used, the field computations for determining the deflection angles to the points on the curve beyond the intermediate transit point are relatively simple and these values are easily set on the vernier, as they rarely contain decimal parts of a minute and often are in half-degrees or whole degrees.

32. Laying Out Curve by Continuous Deflection Angles. In order to explain the fundamental principles underlying the method of laying out a curve by continuous deflection angles, the typical curve shown in Fig. 12 is considered. The line GH is tangent to the curve at the intermediate transit point E, and the line EI is the prolongation of the chord BE. If F is any point on the curve beyond the point E, the deflection angle VBF from the tangent at the P. C. to the point F is equal to $VBE+EBF$. Also, the angle IEF is equal to $IEG+GEF$. But, the angles VBE and HEB are equal, because each is measured by one-half of the arc BE; and the angles HEB and IEG are also equal, because they are vertical angles. Hence, it follows that IEG is equal to VBE. Moreover, EBF and GEF are equal angles, because each is measured by one-half of the arc EF. Therefore, $IEG+GEF=VBE+EBF$, or $IEF=VBF$. In other words, the angle between the prolongation of the chord BE and the chord EF is equal to the deflection angle between the tangent at the P. C. and the chord BF. Hence, with the transit at E, the point F may be located by backsighting to B with the telescope inverted and the vernier reading zero; plunging the telescope back to normal so that it points along the line EI; unclamping the upper plate and setting the vernier to read the deflection angle to the point F with respect to the tangent at the P. C.; and measuring the proper length of the chord EF. All other points on the curve that are visible from E may be located from that point by turning off the deflection angles previously computed for those points for location from the P. C.

33. Where the curve is quite long or the ground is very irregular, it may be necessary to set up the transit at more than one intermediate point on the curve. For example, it may be assumed that, in laying out the curve BC in Fig. 12, it is necessary to set up the transit not only at B and E but also at J, in order to locate the subsequent points on the curve. With the transit at J, the line of sight may be brought into the chord JK by backsighting to E with the telescope inverted and the vernier set at the deflection angle VBE from the tangent at the P. C. to E, plunging the telescope back to normal, and then setting the vernier to read the deflection angle VBK from the tangent at the P. C. to K. This may be proved as follows:

In the illustration, the line LM is tangent to the curve at J, and the line JN is the prolongation of the chord EJ. The angle MJE between the tangent LM at J and the chord JE is equal to the angle GEJ and is also equal to the angle LJN between the tangent LM and the prolongation JN of the chord EJ. Therefore $GEJ=LJN$. Also, $GEJ=IEJ-IEG$. But, $IEJ=VBJ$, and $IEG=VBE$; hence, $LJN=VBJ-VBE$. Finally, since $NJK=LJN+LJK$, and $LJK=JBK$, it follows that $NJK=VBJ-VBE+JBK=VBJ+JBK-VBE$, or $NJK=VBK-VBE$. In other words, the angle between the prolongation JN of the chord EJ and the chord JK is equal to the difference between the deflection angles to K and E from the tangent at the P. C.

34. In general, the transit operations to be performed in laying out a curve by continuous deflection angles with the transit at any intermediate point may be outlined as follows: *Backsight to any previously established point on the curve with the telescope inverted and the vernier set at the deflection angle from the tangent at the P. C. to the point of backsight. Plunge the telescope back to normal and set the vernier to read the deflection angle from the tangent at the P. C. to the point to be located.*

In order that the points on the curve beyond a second intermediate transit point may be correctly located, it is essential to set the vernier reading for the backsight on the proper side of the zero mark. When the method of continuous deflection angles is used, the vernier readings for an entire curve are on the same side of the zero mark.

EXAMPLE.—In laying out a 4° curve to the left, whose P. C. is at Sta. 34+23.08 and whose P. T. is at Sta. 43+83.08, the transit is to be set up at Stas. 38 and 41 and the method of continuous deflection angles is to be used. Compute the deflection angle to each station, and describe the field work for locating Stas. 39 to 43, inclusive.

Solution.—The procedure in this case is indicated in Fig. 13. The deflection angles from the tangent at the P. C. to every station on the curve will be computed before any field work is begun. They will be as follows:

Sta.	Deflection Angle	
35	$0.3\,c'D = 0.3 \times 76.92 \times 4 = 92.3'$ or	$1°\ 32.3'$
36	$1°\ 32.3' + 2° =$	$3°\ 32.3'$
37	$3°\ 32.3' + 2° =$	$5°\ 32.3'$
38	$5°\ 32.3' + 2° =$	$7°\ 32.3'$
39	$7°\ 32.3' + 2° =$	$9°\ 32.3'$
40	$9°\ 32.3' + 2° =$	$11°\ 32.3'$
41	$11°\ 32.3' + 2° =$	$13°\ 32.3'$
42	$13°\ 32.3' + 2° =$	$15°\ 32.3'$
43	$15°\ 32.3' + 2° =$	$17°\ 32.3'$
43+83.08, P. T.	$17°\ 32.3' + \frac{1}{2} \times 0.8308 \times 4° =$	$19°\ 12'$

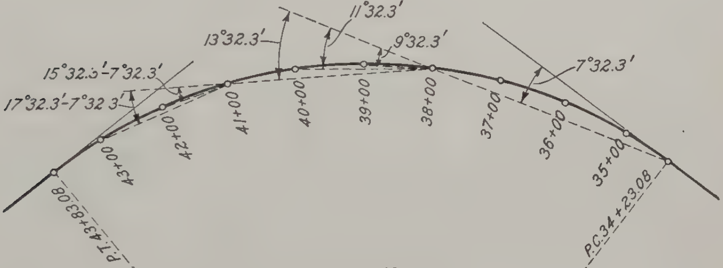

Fig. 13

With the transit at the P. C., Stas. 35 to 38, inclusive, are established on the ground. Then the instrument is moved to Sta. 38. Since a backsight is to be taken to the P. C. and the deflection angle to the P. C. is zero, the first operation at Sta. 38 is to set the vernier to read zero. Then, with the telescope inverted, the line of sight is directed to the P. C. and the lower plate is clamped. To locate Sta. 39, the telescope is plunged back to normal, the upper clamp is loosened, and the telescope is turned counter-clockwise—in this case, the curve is to the left—until the vernier reading is equal to the deflection angle from the tangent at the P. C. to Sta. 39, or 9° 32.3′. Then the length of the chord between Stas. 38 and 39 is measured along the line of sight and Sta. 39 is established at the end of the measurement. Sta. 40 is located by loosening the upper clamp, setting the vernier to the deflection angle for that station, or 11° 32.3′, and measuring the chord from Sta. 39 to Sta. 40 so that the end of the measurement comes in the line of sight from the transit at Sta. 38. Similarly, Sta. 41 is established by setting the vernier to read 13° 32.3′ and measuring the proper chord length from Sta. 40.

When the transit is moved to Sta. 41, the backsight is taken to Sta. 38. The vernier reading for the backsight is in this case the deflection angle to Sta. 38 from the tangent at the P. C., or 7° 32.3′. This setting is on the same side of the zero mark as the settings for the other deflection angles. With the vernier properly set and the telescope inverted, the line of sight is directed to Sta. 38. Then, the telescope is plunged back to normal, the upper plate is unclamped, and Sta. 42 is located by setting the vernier to read the deflection angle to that station from the tangent at the P. C., or 15° 32.3′, and measuring the proper chord length from Sta. 41. Since the vernier is set to read 7° 32.3′ when the backsight is taken along the chord from Sta. 41 to Sta. 38, the angle actually turned from that chord to the chord from Sta. 41 to Sta. 42 is 15° 32.3′−7° 32.3′, as indicated in Fig. 13. Also, Sta. 43 is established by setting the vernier to 17° 32.3′ and measuring the chord from Sta. 42.

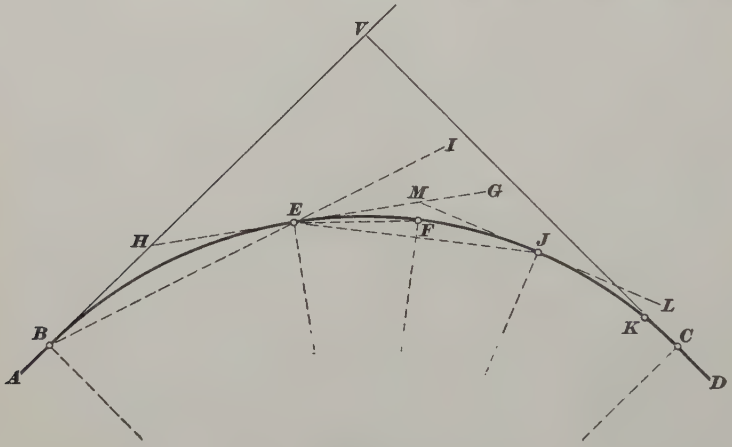

FIG. 14

35. Deflection Angles From Auxiliary Tangent.—In order to use the method of laying out a curve by deflection angles from an auxiliary tangent, it is first necessary to determine the deflection angles from the auxiliary tangent to other points on the curve. In Fig. 14 the tangents AB and CD, intersecting at V, are connected by a simple curve BC. It is assumed that the transit has to be set up at the point E, and that the point F is to be located by the deflection angle GEF from the tangent GH at E.

The deflection angle *GEF* is measured by one-half of the arc *EF*, and is, therefore, equal to one-half of the central angle subtended by that arc. Thus, a formula similar to that in Art. 22 may be applied for finding the deflection angle from the tangent at any point on a curve to any other point, or

$$d_1=\frac{s_1D}{2}$$

in which d_1 = deflection angle from tangent at any point on curve to any other point;
s_1 = difference, in stations of 100 feet, between station numbers of two points;
D = degree of curve.

After the deflection angle to the first 100-foot station beyond any intermediate transit point on the curve has been determined by means of the preceding formula, the deflection angle to each subsequent station up to and including the next transit point may be found by applying the principle of Art. 23, or by adding one-half of the degree of curve to the deflection angle for the preceding station. If there is more than one intermediate transit point on the same curve, the computation of deflection angles starts anew at each such point.

As a check on the calculations, the sum of the deflection angles to the intermediate transit points on the curve and to the P. T. should equal one-half of the angle of intersection between the tangents at the P. C. and P. T.

EXAMPLE.—If the curve in the example of Art. 34 is to be laid out by the method of deflection angles from an auxiliary tangent and the same intermediate set-ups are required, what should be the deflection angle to each station on the curve?

SOLUTION.—The deflection angles up to Sta. 38 are found as in the preceding article. They are:

STA.	DEFLECTION ANGLE
35	$0.3\,c'D=0.3\times76.92\times4=92.3'$ or $1^\circ\ 32.3'$
36	$1^\circ\ 32.3'+2^\circ=3^\circ\ 32.3'$
37	$3^\circ\ 32.3'+2^\circ=5^\circ\ 32.3'$
38	$5^\circ\ 32.3'+2^\circ=7^\circ\ 32.3'$

The deflection angles to Stas. 39, 40, and 41 are referred to the tangent at Sta. 38. For Sta. 39 the value of s_1 in the formula of this article is 1 and the required deflection angle is

$$d_1=\frac{s_1D}{2}=\frac{1\times 4}{2}=2°$$

By Art. 23, the deflection angle to Sta. 40 is

$$2°+\tfrac{1}{2}\times 4°=4°$$

and that to Sta. 41 is

$$4°+\tfrac{1}{2}\times 4°=6°$$

The deflection angles from the tangent at Sta. 41 to Stas. 42 and 43 and to the P. T. are as follows:

Sta. 42, $\frac{s_1D}{2}=\frac{1\times 4}{2}=2°$

Sta. 43, $2°+\tfrac{1}{2}\times 4°=4°$

Sta. 43+83.08, P. T., $4°+\tfrac{1}{2}\times 0.8308\times 4=5°\ 39.7'$

As a check, it is desirable to find the sum of the deflection angles to Stas. 38 and 41 and to the P. T. Thus,

$$7°\ 32.3'+6°+5°\ 39.7'=19°\ 12'$$

which is equal to $\tfrac{1}{2}I$ or $\tfrac{1}{2}\times 38°\ 24'$.

36. Laying Out Curve by Deflection Angles From Auxiliary Tangent.—The procedure in laying out a curve by deflection angles from an auxiliary tangent is similar to that described for the method by continuous deflection angles, but different vernier readings are used in the two methods.

If, for example, the transit is set up at the intermediate point *E*, Fig. 14, and it is desired to make the vernier read the deflection angle to *F* from the tangent *GH* at *E* when the telescope is directed along the chord *EF*, the vernier reading must be zero when the line of sight lies in the tangent *GH*. This zero reading is achieved as follows: First the inverted telescope is directed to *B* with the vernier set to the value of the angle *HEB*, which is equal to the deflection angle *VBE* from the tangent at the P. C. to the point *E*. Then, the telescope is plunged back to normal, so that the line of sight will be directed along the line *EI*, which is the prolongation of the chord *BE*. Finally, since angle *GEI* is equal to angle *HEB*, the line of sight is brought into the tangent *EG* by setting the vernier to read zero. Thus, with the transit at *E*, the point *F* may be located by taking a backsight on the P. C. at B with the vernier reading the value of the angle VBE and the telescope inverted; plunging the telescope back to normal; setting the vernier to read the angle GEF on the opposite side of the zero mark, so that the line of sight will be directed along the chord EF; and measuring the required length of the chord EF along the line of sight. From a study of the illustration, it will be seen that the vernier readings for the backsight to B and for the foresight along the chord EF must be on opposite sides of the zero point.

In case it is necessary to set up the transit at another intermediate point on the curve, as J, and the point K is to be located by the deflection angle from the auxiliary tangent LM at J, the procedure would be as follows: First, with the vernier reading equal to the angle MJE, or to the deflection angle MEJ from the tangent GH to J, and with the telescope inverted, a backsight is taken to E. Then, the telescope is plunged back to normal, and the vernier is turned through zero and set at the deflection angle LJK, the line of sight thus being brought in the chord JK. As at the first intermediate point, the readings for the backsight and the foresight must be on opposite sides of the zero mark.

37. From the explanations in the preceding article, the general method of performing the transit operations at an intermediate point on a curve when the deflection angles are to be referred to the tangent at that point may be outlined as follows: *Backsight to any previously established point on the curve with the telescope inverted and the vernier set to read the deflection angle to the transit point from the tangent at the point of backsight. Then plunge the telescope back to normal, unclamp the upper plate, and, after passing the zero mark, set the vernier to read the deflection angle from the tangent at the transit point to the point to be located.*

If the curve turns to the left, the vernier reading for the backsight should be obtained by setting the vernier roughly to zero and then rotating the telescope clockwise. On the other hand, if the curve turns to the right, the vernier setting should be obtained by rotating the telescope counter-clockwise from the zero position. After the telescope has been plunged back to normal, the proper deflection angle for the foresight is set on the vernier on the opposite side of zero without stopping at the zero mark. The chord measurements are made in the same way as for the method of continuous deflection angles.

EXAMPLE.—For the curve in the example of Art. 35, find the vernier settings (*a*) for the backsight from Sta. 38 to the P. C. and (*b*) for the backsight from Sta. 41 to Sta. 38.

Station	Point	Description of Curve	Deflection Angle	Magnetic Bearing	
				Observed	Calculated
45					
44				N 84° E	N 84° 11′ E
+83.08	P.T.		5°39.7′		
43			4°00′		
42		I = 38°24′	2°00′		
41	I.P. on ℄	D = 4° L	6°00′		
40		R = 1,432.69′	4°00′		
39	39 + 22 V	T = 498.92′	2°00′		
38	I.P. on ℄	L = 960.00′	7°32.3′		
37			5°32.3′		
36			3°32.3′		
35			1°32.3′		
+23.08	P.C.				
34					
33				S 57°30′ E	S 57°25′ E

FIG. 15

SOLUTION.—(*a*) When taking the backsight from Sta. 38 to the P. C., the telescope is inverted and the required vernier reading is the deflection angle to Sta. 38 from the tangent at the P. C., or 7° 32.3′. Since the curve in this case turns to the left, the setting is made by rotating the telescope in a clockwise direction from the position it would have when the vernier reads zero.

To locate Stas. 39, 40, and 41, the telescope is plunged back to normal and the vernier is turned beyond zero to the deflection angles of 2°, 4°, and 6° in succession.

(*b*) The required vernier reading for the backsight from Sta. 41 to Sta. 38 is the deflection angle to Sta. 41 from the tangent at Sta. 38. From Art. 35, this angle is 6°. As for the backsight from Sta. 38 to the P. C., the vernier setting at Sta. 41 is obtained by rotating the telescope in a clockwise direction from its position for a vernier reading of zero.

38. Field Notes for Curves With Intermediate Transit Points.—In Fig. 15 is shown a typical method of recording the field notes for a curve that was laid out by the method of deflection angles from an auxiliary tangent. These notes are for a curve whose degree is based on a 100-foot chord, but the method is exactly the same for a curve whose degree is determined by a 100-foot arc. Each transit point on the curve is indicated by the abbreviation I. P. on ℄ (meaning instrument point on center line) in the second column. The values of the deflection angles in the fourth column are computed as in Art. 35. In other respects these notes are similar to those for a curve that is laid out entirely from the P. C.

The field notes for a curve that is located from intermediate points by continuous deflection angles are, in the main, similar to those shown in Fig. 10 or 11. However, instrument points on the curve are identified by the notation, I. P. on ℄, as in the second column in the form of notes shown in Fig. 15.

EXAMPLES FOR PRACTICE

1. Two tangents that intersect at an angle of 29° 50′ are joined by a 3° 30′ curve. The P. C. is at Sta. 95+76.30 and the P. T. is at Sta. 104+28.68 and, in order to lay out the curve in the field, it is necessary to set up at Stas. 98 and 103. If the method of continuous deflection angles is used, what should be (*a*) the deflection angle to each 100-foot station on the curve and (*b*) the vernier reading when the transit is set up at Sta. 103 and the backsight to Sta. 98 is taken?

Ans. (*a*) 24.9′, 2° 9.9′, 3° 54.9′, 5° 39.9′, 7° 24.9′, 9° 9.9′, 10° 54.9′, 12° 39.9′, 14° 24.9′
(*b*) 3° 54.9′

2. In laying out a 7° curve whose P. C. is at Sta. 23+09.53, it is found necessary to set up the transit at Sta. 27. If the method of deflection angles from an auxiliary tangent is to be used, what should be the vernier reading (*a*) when the backsight from Sta. 27 to the P. C. is taken, and (*b*) when the foresight to Sta. 28 is taken?

Ans. (*a*) 13° 40′
(*b*) 3° 30′

LAYING OUT CURVE WITH TRANSIT AT P. T.

39. When the entire curve is visible from the P. T., the intermediate stations on the curve may be conveniently located with the transit at the P. T. instead of at the P. C. Thus, it is assumed that the curve *ACDEFB*, Fig. 16, is entirely visible from the P. T. at *B* and the stations at *C*, *D*, *E*, and *F* are located with the transit at point *B*. Since the angle *ABC* is an inscribed angle, it is measured by one-half of the arc *AC* and is equal to one-half of the central angle *AOC*. The deflection angle *VAC* from the tangent at the P. C. to point *C* is also one-half of the central angle *AOC*, and therefore the angle *ABC* is equal to the deflection angle *VAC*. Similarly, it can be shown that the angles *ABD*, *ABE*, and *ABF* are equal, respectively, to the deflection angles from the tangent at the P. C. to points *D*, *E*, and *F*. Consequently, if the transit is set at the P. T. and the telescope is directed to the P. C. with the vernier reading zero, the intermediate stations on the curve may be located by turning off in succession the regular deflection angles with respect to the tangent at the P. C. and measuring the chord lengths from station to station in the usual manner. Although the transit is at the P. T., the stations are located by proceeding from the P. C. toward the P. T. and the vernier settings

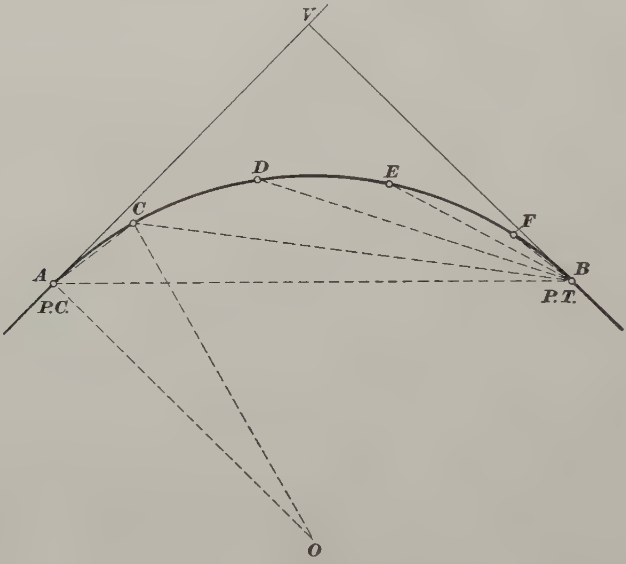

FIG. 16

are made exactly as if the transit were at the P. C. After all the stations have been established, the work may be checked by setting the vernier at $\frac{I}{2}$ and sighting towards V. If the transit work is correct, the point V should be in the line of sight.

40. The method described in the preceding article is used to a considerable extent because it is preferable for several reasons to the method with the transit at the P. C. First, if the P. C. and the P. T. are located from the P. I., the transit can be moved directly from the P. I. to the P. T., and it need not be set up at the P. C. Thus, the trouble of carrying the transit from the P. I. to the P. C. and then to the P. T. is avoided, and the transit is in the same position for laying out the curve as for continuing the line along the extension of the tangent through the P. T. Second, the long sights are taken immediately after the transit is set up and before settlement and sunlight have disturbed the adjustment of the instrument. Third, since the successive lines of sight along the chords keep getting nearer to the curve, accumulated errors in chaining do not affect the alinement of the curve so much. Therefore, the established position of the curve on the ground is likely to be more nearly correct.

TANGENT OFFSETS

41. Use of Tangent Offsets.—In railroad or highway work, it often happens that an obstacle in the line of sight of the transit prevents the use of the method of deflection angles for locating a single point or several points on a curve. The stations that cannot be established by deflection angles are then generally located from the P. C. or the P. T. by measuring distances along and perpendicular to the tangent at the P. C. or the P. T. These perpendicular distances are commonly called tangent offsets, or sometimes tangent deflections, and it is said that the points are located by tangent offsets. When a transit is not available, the method of tangent offsets is sometimes used to lay out a complete curve on the ground.

42. Calculations for Tangent Offsets.—In Fig. 17 the tangents AB and CD are connected by the simple curve BC, the center of which is at O. To locate any point, such as E, on the curve by the method of offsets from the tangent at the P. C., it is necessary to calculate and measure both the distance BE', or x, along that tangent from the P. C. to the foot of the perpendicular EE', and the tangent offset EE', or y. The distances x and y for a point are sometimes called the coordinates of the point. In order to derive formulas for x and y, the line EE'' is drawn parallel to the tangent AV until it intersects the radius OB. Then, in the triangle EOE'', $EE''=OE \sin EOE''$; or since $EE''=BE'$ and OE is the radius R of the curve,

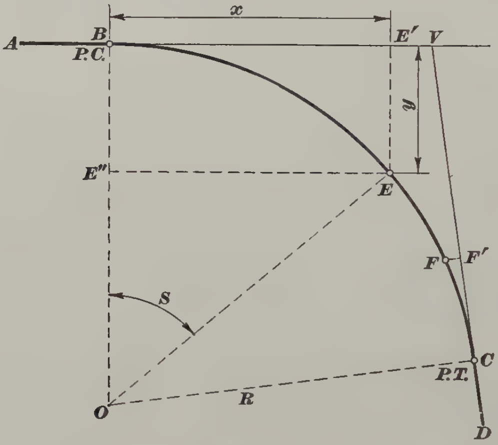

Fig. 17

$$x = R \sin S \qquad (1)$$

in which x = distance along the tangent from the P. C. to the foot of the perpendicular from the point to be located;

R = radius of curve;

S = total central angle between the radius to the P. C. and the radius to the point to be located.

Also, $E'E=BE''=OB-OE''$ and, in the triangle EOE'', $OE''=OE \cos EOE''$. Hence, since $OB=OE=R$, the tangent offset $y=R-R \cos S$, or

$$y = R\,(1-\cos S) \qquad (2)$$

The difference between 1 and the cosine of an angle is the trigonometric function of the angle that is known as the versine or versed sine, the abbreviation for which is *vers*. Thus, vers $S = 1 - \cos S$ and

$$y = R \text{ vers } S \qquad (3)$$

For any given point on the curve, the angle S is found by multiplying the degree of curve by the difference between the station numbers of the P. C. and the point to be located on the curve.

EXAMPLE 1.—In laying out a 2° railroad curve, whose degree is based on a 100-foot chord and whose P. C. is at Sta. 21+16, it is desired to locate Sta. 25+00 by the method of tangent offsets. What are the coordinates of that station with respect to the tangent at the P. C.?

SOLUTION.—From Table I, the radius of the curve is $R = 2{,}864.93$ ft. Since the degree of curve is 2° and the distance from the P. C. at Sta. 21+16 to Sta. 25+00 is 3.84 stations, the central angle between the radii to these two points is $S = 2 \times 3.84 = 7.68°$ or $7° 40.8'$. By formulas 1 and 2, the required coordinates are

$$x = R \sin S = 2{,}864.93 \sin 7° 40.8' = 382.87 \text{ ft.} \quad \text{Ans.}$$
$$y = R\,(1 - \cos S) = 2{,}864.93 \times (1 - \cos 7° 40.8') = 25.70 \text{ ft.} \quad \text{Ans.}$$

EXAMPLE 2.—The P. C. of a 6° highway curve, whose degree is based on an arc of 100 feet, is at Sta. 8+20. Calculate the coordinates of Sta. 10+50.

SOLUTION.—The distance from Sta. 8+20 to Sta. 10+50 is 2.3 stations and the central angle between the radii to those two points on a 6° curve is $6 \times 2.3 = 13.8°$ or $13° 48'$. Also, the radius of a 6° curve is, from Table II, 954.93 ft. Then, by formulas 1 and 2,

$$x = R \sin S = 954.93 \sin 13° 48' = 227.78 \text{ ft.} \quad \text{Ans.}$$
$$y = R\,(1 - \cos S) = 954.93 \times (1 - \cos 13° 48') = 27.57 \text{ ft.} \quad \text{Ans.}$$

43. If desired, points near the P. T. can be located by distances along and perpendicular to the tangent at the P. T. For example, the point F, Fig. 17, may be located by the distance CF' along the tangent CV and the offset $F'F$ perpendicular to that tangent. The method of computing coordinates referred to the tangent at the P. T. is exactly the same as that used when the coordinates are referred to the tangent at the P. C., but the angle S in formulas 1 and 2 of the preceding article is measured from the radius to the P. T.

EXAMPLE.—If the P. T. of the 6° highway curve in example 2 of Art. 42 is at Sta. 12+65, what is the tangent offset for Sta. 12 with respect to the tangent at the P. T.?

SOLUTION.—The central angle between the radii to Sta. 12+00 and the P. T. is $6° \times 0.65 = 3.9°$ or 3° 54′. Then, by formula 2, Art. 42,

$$y = R\ (1 - \cos S) = 954.93 \times (1 - \cos 3°\ 54') = 2.22 \text{ ft.} \quad \text{Ans.}$$

44. Special Method for Curve Whose Degree is Based on 100-Foot Chord.—The formulas of Art. 42 may be used for determining the coordinates of any point on a simple curve. However, if the degree of curve is taken as the angle subtended by a 100-foot chord and the entire curve is to be located by the method of tangent offsets, it is generally convenient to use those formulas only at the last station—as a check on the work—and to compute the coordinates of each intermediate station by the following method.

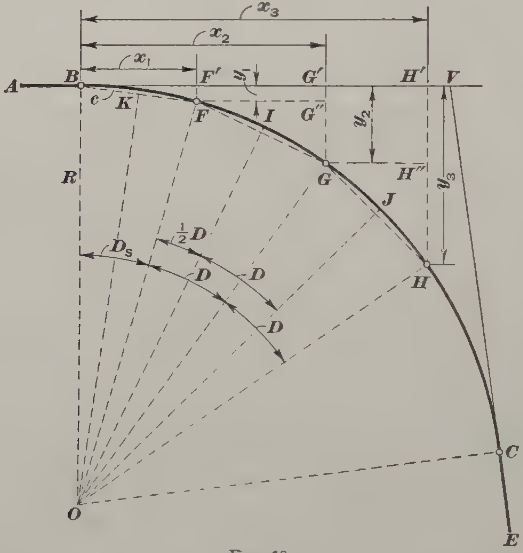

FIG. 18

In Fig. 18 the tangents AB and CE are joined by the curve BC whose center is at O; the points F, G, and H are at 100-foot stations on the curve; and the lines FF', GG', and HH' are drawn perpendicular to the tangent AV at the P. C. It is assumed that the distance BF is less than 100 feet, and the central angle BOF is designated as D_s. The coordinates of F, which are $x_1 = BF'$ and $y_1 = F'F$, may be readily computed by solving the right triangle FBF'. In this triangle the hypotenuse BF is the subchord c from the P. C. to the first full station on the curve, and the angle $F'BF$ is the deflection angle from the tangent at the P. C. to the first full station and is, therefore, equal to $\frac{1}{2}D_s$. Hence,

$$x_1 = c \cos \tfrac{1}{2}D_s \qquad (1)$$

and

$$y_1 = c \sin \tfrac{1}{2}D_s \qquad (2)$$

in which x_1 and y_1 are coordinates of first full station on curve, in feet;

c = length of subchord from P. C. to first station on curve, in feet;

D_s = central angle between radii to P. C. and first station.

These formulas also apply when the P. C. is at a 100-foot station. In that case, $D_s = D$ and c is 100 feet.

In order to derive expressions for the coordinates of the point G, namely, $x_2 = BG'$ and $y_2 = G'G$, the line FG'' is drawn parallel to AV, and the line OI is drawn perpendicular to the chord FG, thus bisecting that chord and its arc. Obviously, $BG' = BF' + F'G' = BF' + FG''$, and $G'G = G'G'' + G''G = F'F + G''G$. The distances BF' and $F'F$ are the coordinates of F, and the distances FG'' and $G''G$ can be found by solving the right triangle FGG''. The angle $G''FG$ is equal to the angle BOI, because the sides of one angle are perpendicular to the sides of the other; hence, $FG'' = FG \cos G''FG = FG \cos BOI$, and $G''G = FG \sin BOI$. Then, $x_2 = x_1 + 100 \cos BOI$, and $y_2 = y_1 + 100 \sin BOI$.

Similarly, to find the coordinates $x_3 = BH'$ and $y_3 = H'H$ of the point H, the line GH'' is drawn parallel to the tangent AV and the line OJ is drawn perpendicular to the chord GH. Then, $BH' = BG' + G'H' = BG' + GH''$, and $H'H = H'H'' + H''H = G'G + H''H$. But $BG' = x_2$ and $G'G = y_2$; also, since the angles $H''GH$ and BOJ are equal, $GH'' = GH \cos BOJ$ and $H''H = GH$

sin BOJ. Hence, the coordinates are $x_3 = x_2 + 100 \cos BOJ$ and $y_3 = y_2 + 100 \sin BOJ$.

In general, the coordinates of any 100-foot station on a curve whose degree is determined by a 100-foot chord can be found from the coordinates of the preceding station by the following formulas:

$$x = x' + 100 \cos A \qquad (3)$$

$$y = y' + 100 \sin A \qquad (4)$$

in which x and y are coordinates of 100-foot station to be located, in feet;

x' and y' are coordinates of preceding 100-foot station on curve, in feet;

A = angle between radius to P. C. and radius to point midway between the station to be located and the preceding 100-foot station.

The value of the angle A for the second 100-foot station on the curve is equal to $D_s + \frac{1}{2}D$, and the angle A for each subsequent 100-foot station is found by adding D to the value of A for the preceding station. Thus, in Fig. 18, the angle BOI is equal to $D_s + \frac{1}{2}D$, and the angle BOJ is equal to $BOI + D$.

EXAMPLE.—If the P. C. of a 6° curve, whose degree is determined by a 100-foot chord, is at Sta. 8+20, what are the values of the coordinates of Stas. 9, 10, and 11?

SOLUTION.—The length of the subchord from the P. C. to Sta. 9 is $c = 80$ ft., and the central angle subtended by that subchord is $D_s = 6° \times 0.8 = 4.8°$ or 4° 48′. Hence, by formulas 1 and 2, the coordinates of Sta. 9 are

$$x_1 = c \cos \tfrac{1}{2}D_s = 80 \cos 2° \, 24' = 79.93 \text{ ft.} \quad \text{Ans.}$$

$$y_1 = c \sin \tfrac{1}{2}D_s = 80 \sin 2° \, 24' = 3.35 \text{ ft.} \quad \text{Ans.}$$

The value of A to be used in formulas 3 and 4 for Sta. 10 is $D_s + \frac{1}{2}D = 4°48' + \frac{1}{2} \times 6° = 7° \, 48'$, and the values of x' and y' are the coordinates of Sta. 9. Thus, for Sta. 10,

$$x = x' + 100 \cos A = 79.93 + 100 \cos 7° \, 48' = 179.01 \text{ ft.} \quad \text{Ans.}$$

$$y = y' + 100 \sin A = 3.35 + 100 \sin 7° \, 48' = 16.92 \text{ ft.} \quad \text{Ans.}$$

For Sta. 11, the angle A, which is found by adding D to the angle for Sta. 10, is 7° 48′ + 6° = 13° 48′, and the coordinates are

$$x = 179.01 + 100 \cos 13° \, 48' = 276.12 \text{ ft.} \quad \text{Ans.}$$

$$y = 16.92 + 100 \sin 13° \, 48' = 40.77 \text{ ft.} \quad \text{Ans.}$$

45. Geometrical Formulas for Tangent Offsets.—The tangent offset to any point on a simple curve may also be computed by the principles of geometry. In Fig. 18, *BF* is a chord of any length, *FF′* is the tangent offset for the point *F*, and *OK* is the perpendicular bisector of the chord. Then, since the angles *F′BF* and *BOK* are equal, the right triangles *BFF′* and *BOK* are similar and

$$OB:BK = BF:F'F$$

If the radius *OB* is denoted by *R*, the length of the chord *BF* by *c*, and the tangent offset *F′F* by *y*, this relation becomes

$$R:\tfrac{1}{2}c = c:y$$

or

$$y = \frac{c^2}{2R} \qquad (1)$$

This formula is correct for any point. Hence, if y_0 represents the offset for a point at a chord distance of 100 feet from the P. C., then

$$y_0 = \frac{100^2}{2R}$$

and

$$\frac{y}{y_0} = \frac{c^2}{2R} \div \frac{100^2}{2R} = \frac{c^2}{100^2}$$

or

$$y = y_0 \times \left(\frac{c}{100}\right)^2 \qquad (2)$$

in which y = tangent offset for any point on curve, in feet;
y_0 = tangent offset for point at chord distance of 100 feet from extremity of curve, in feet;
c = length of chord from extremity of curve to point under consideration, in feet.

Formula 2 is often applied where the degree of curve is based on a chord of 100 feet. Values of the tangent offset for a point at a chord distance of 100 feet from the P. C. or P. T. are given for various degrees of curve in the column of Table I that is headed Tangent Offset. Thus, for a 3° curve the tangent offset for a point 100 feet from the P. C. or P. T. is found from the table to be 2.618 feet.

EXAMPLE.—Find, by the use of formula 2, the tangent offset at Sta. 9 of the curve in the example of Art. 44.

SOLUTION.—From Table I, the value of y_0 for a 6° curve is 5.234 ft.; and the length of the chord from the P. C. to Sta. 9 is $c=80$ ft. Hence, the required tangent offset is

$$y=y_0\times\left(\frac{c}{100}\right)^2=5.234\times0.8^2=3.35\text{ ft.}\quad\text{Ans.}$$

46. Field Location by Tangent Offsets.—After the distances along the tangent and the tangent offsets have been computed for the various points to be located on the curve, the first step in laying out the curve is to set stakes at the required points along the tangent, as at F', G', and H' in Fig. 18. Where there are no obstructions on the curve, each station along the curve is best located by measuring the offset from the tangent and the chord from the preceding station, and establishing the intersection of these two distances. Thus, the point F is set at the intersection of two arcs, one having F' as a center and the tangent offset $F'F$ as a radius and the other having B as a center and the chord length BF as a radius. Similarly, the point G is established at the intersection of arcs swung from G' and F with radii of $G'G$ and FG, respectively.

In case the chord from the preceding station to the point to be located cannot be conveniently measured, the tangent offset for the point is laid off at right angles to the tangent. Thus, the point F, Fig. 18, may be located by measuring the distance BF' along the tangent AV, erecting a perpendicular to the tangent at F', and laying off the tangent offset $F'F$ along that perpendicular.

EXAMPLES FOR PRACTICE

1. The P. C. of an 8° curve, whose degree is based on an arc of 100 feet, is at Sta. 11+19.2. Calculate the coordinates of Sta. 15+00.

$$\text{Ans.}\begin{cases}x=363.11\text{ ft.}\\y=98.87\text{ ft.}\end{cases}$$

2. The P. C. of a 4° curve, whose degree is based on a 100-foot chord, is at Sta. 3+37.4. Calculate the coordinates of Stas. 4, 5, and 6.

$$\text{Ans.}\begin{cases}\text{Sta. 4—}x=62.58\text{ ft., }y=1.37\text{ ft.}\\\text{Sta. 5—}x=162.27\text{ ft., }y=9.22\text{ ft.}\\\text{Sta. 6—}x=261.17\text{ ft., }y=24.01\text{ ft.}\end{cases}$$

3. Find the tangent offset at Sta. 4 of the curve in the preceding example by the use of formula 2, Art. 45. Ans. 1.37 ft.

ORDINATES FROM LONG CHORD

47. General Procedure.—Another method of locating points on a curve without the use of a transit consists in measuring calculated distances along and perpendicular to the long chord, which is the chord joining the P. C. and P. T. In Fig. 19, the two tangents *AB* and *EF* are connected by the simple curve *BE*, the long chord of which is the straight line from the P. C. at *B* to the P. T. at *E*. Any station on the curve, as *G*, *H*, *J*, *K*, *L*, *M*, or *N*, may be located by first laying off along the long chord the computed distance from the P. C. or P. T. to the foot of the perpendicular dropped from the respective station to the long chord, and then measuring along that perpendicular the computed ordinate from the chord to the station. Thus, station *G* would be established by laying off the computed dis-

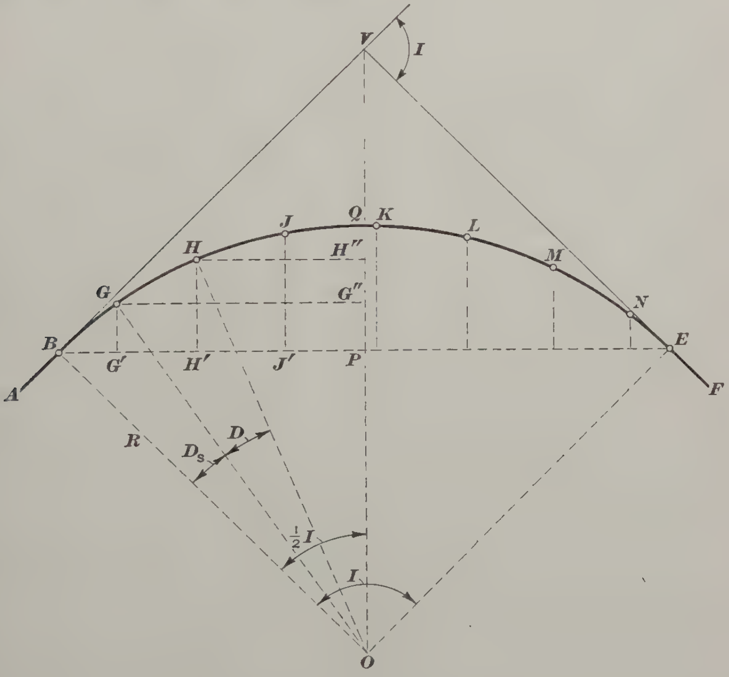

FIG. 19

tance BG' on the long chord, setting a temporary stake at G', erecting a perpendicular to the long chord at G', and measuring the computed ordinate $G'G$ along that perpendicular. Similarly, station H may be located by measuring the distance BH' along the chord BE and the ordinate $H'H$ on the perpendicular erected at H'.

This method is known as the method of ordinates from the long chord. It is best adapted for bending rails to the proper curvature, but occasionally it is found convenient for locating the stations on a curve whose degree is based on an arc of 100 feet, where neither the method of deflection angles nor the method of tangent offsets is suitable.

48. Calculation of Distances.—If it is assumed that the P. C. of the curve in Fig. 19 is at a plus station, as is usually the case, the central angle BOG is less than the degree of curve D; this angle will be denoted by D_s. The distances required for locating the various stations on the curve may be determined as follows: The line OV from the center point O to the vertex V bisects the central angle I, and angle $BOV = \frac{1}{2}I$. Also, the line OV is perpendicular to the long chord BE and bisects that chord at P. If GG'' is drawn parallel to BE, it is evident that $GG''=G'P$, and $BG'=BP-GG''$; also, $G'G = PG''=OG''-OP$. If R represents the radius of the curve, $BP=R \sin \frac{1}{2}I$, $OP=R \cos \frac{1}{2}I$, $GG''=R \sin (\frac{1}{2}I-D_s)$, and $OG''=R \cos (\frac{1}{2}I-D_s)$. Therefore,

$$BG'=R\left[\sin \tfrac{1}{2}I-\sin (\tfrac{1}{2}I-D_s)\right]$$
$$G'G=R\left[\cos (\tfrac{1}{2}I-D_s)-\cos \tfrac{1}{2}I\right]$$

If HH'' is made parallel to the long chord BE, then $BH' = BP-HH''=R \sin \frac{1}{2}I-R \sin (\frac{1}{2}I-D_s-D)$ and $H'H=OH'' -OP=R \cos (\frac{1}{2}I-D_s-D)-R \cos \frac{1}{2}I$. Hence,

$$BH'=R\left[\sin \tfrac{1}{2}I-\sin (\tfrac{1}{2}I-D_s-D)\right]$$
$$H'H=R\left[\cos (\tfrac{1}{2}I-D_s-D)-\cos \tfrac{1}{2}I\right]$$

For any station between the P. C. and the middle point of the curve, the distance from the P. C. to the foot of the perpendicular from the station to the long chord is

$$z=R\left[\sin \tfrac{1}{2}I-\sin (\tfrac{1}{2}I-S)\right] \qquad (1)$$

and the perpendicular distance, or ordinate, from the long chord to the station is

$$o = R\left[\cos\left(\tfrac{1}{2}I - S\right) - \cos\tfrac{1}{2}I\right] \qquad (2)$$

in which z = distance, in feet, measured from the P. C. along the long chord to the foot of the perpendicular to that chord from the station under consideration;
R = radius of curve, in feet;
I = intersection angle;
S = central angle between radii to P. C. and to station to be located;
o = ordinate from long chord to station, in feet.

For a station between the middle point of the curve and the P. T., formulas 1 and 2 can be applied simply by taking z as the distance from the P. T., and S as the angle between the radii to the P. T. and the station. In case all distances along the long chord are to be measured from the P. C., the distance for a station between the middle of the curve and the P. T. may be found by taking the difference between the length of the long chord and the distance along that long chord from the P. T. to the station. The length C_l of the long chord may be computed by the formula

$$C_l = 2R\sin\tfrac{1}{2}I \qquad (3)$$

EXAMPLE.—If the curve in Fig. 19 is a 3° highway curve, whose degree is based on an arc of 100 feet, its P. C. is at Sta. 75+30, and the total central angle is 21° 36′, compute the distances along the long chord and the ordinates (*a*) for Sta. 76, (*b*) for Sta. 77, and (*c*) for Sta. 81.

SOLUTION.—(*a*) In this case, the radius R is, from Table II, 1,909.86 ft. and $\frac{1}{2}I = \frac{1}{2} \times 21°\ 36' = 10°\ 48'$. For Sta. 76, which is at a distance of 70 ft. from the P. C., the value of S in formulas 1 and 2 is $3° \times 0.7 = 2.1°$ or $2°\ 6'$. Hence, the distance along the long chord from the P. C. to the ordinate at Sta. 76 is

$$z = R\left[\sin\tfrac{1}{2}I - \sin\left(\tfrac{1}{2}I - S\right)\right] = 1{,}909.86 \times (\sin 10°\ 48' - \sin 8°\ 42')$$
$$= 68.98 \text{ ft.} \quad \text{Ans.}$$

Also, the ordinate at that station is

$$o = R\left[\cos\left(\tfrac{1}{2}I - S\right) - \cos\tfrac{1}{2}I\right] = 1{,}909.86 \times (\cos 8°\ 42' - \cos 10°\ 48')$$
$$= 11.84 \text{ ft.} \quad \text{Ans.}$$

(*b*) For Sta. 77, $S = 2° 6' + 3° = 5° 6'$. Hence,

$$z = 1{,}909.86 \times (\sin 10° 48' - \sin 5° 42') = 168.18 \text{ ft.} \quad \text{Ans.}$$

and $$o = 1{,}909.86 \times (\cos 5° 42' - \cos 10° 48') = 24.39 \text{ ft.} \quad \text{Ans.}$$

(*c*) The length of the curve is $\frac{21.6}{3} \times 100 = 720$ ft. and the P. T. is, therefore, at Sta. 82+50. The central angle between the radii to the P. T. and to Sta. 81 is 3°×1.5=4.5° or 4° 30′. Hence, the distance from the P. T. to the ordinate at Sta. 81 is, by formula 1,

$$z = 1{,}909.86 \times (\sin 10° 48' - \sin 6° 18') = 148.30 \text{ ft.}$$

By formula 3, the length of the long chord is

$$C_l = 2R \sin \tfrac{1}{2}I = 2 \times 1{,}909.86 \times \sin 10° 48' = 715.74 \text{ ft.}$$

and the required distance along the long chord from the P. C. to Sta. 81 is

$$715.74 - 148.30 = 567.44 \text{ ft.} \quad \text{Ans.}$$

Also, by formula 2, in which $S = 4° 30'$, the ordinate at Sta. 81 is

$$o = 1{,}909.86 \times (\cos 6° 18' - \cos 10° 48') = 22.29 \text{ ft.} \quad \text{Ans.}$$

49. Middle Ordinate.—The distance from the long chord to the curve, measured along the line joining the center of the curve and the vertex, is called the middle ordinate. Thus, in Fig. 19, the middle ordinate is the distance PQ measured along the line OV from the long chord BE to the curve. The middle ordinate is the greatest ordinate from the long chord to the curve. Since $PQ = OQ - OP = R - R \cos \tfrac{1}{2}I$, the middle ordinate may be found by the formula

$$m = R\,(1 - \cos \tfrac{1}{2}I) \qquad (1)$$

or

$$m = R \text{ vers } \tfrac{1}{2}I \qquad (2)$$

in which m = middle ordinate, in feet;

R = radius of curve, in feet;

I = intersection angle.

50. Approximate Ordinates.—When the long chord is not greater than 100 feet in length, it is convenient and sufficiently accurate to use the following approximate method for determining the ordinate to any point on the curve. In Fig. 20, AB is the long chord for the curve ACB, whose center is at O, and CD is the middle ordinate. If OE is drawn perpendicular to the chord AC, the right triangles ACD and AOE are similar, and $\frac{CD}{AC} = \frac{AE}{OA}$. But $AE = \tfrac{1}{2}AC$ and, if the length of the chord

AB is small compared with the radius OA, the length of the chord AC may be assumed to be equal to the distance AD or half of the length of the chord AB. Now, let m represent the middle ordinate; R, the radius of the curve; and c, the length of the chord whose middle ordinate is m. Then,

$$\frac{m}{\frac{1}{2}c}=\frac{\frac{1}{2}c}{R}$$

or

$$m=\frac{c^2}{8R} \qquad \textbf{(1) Approx.}$$

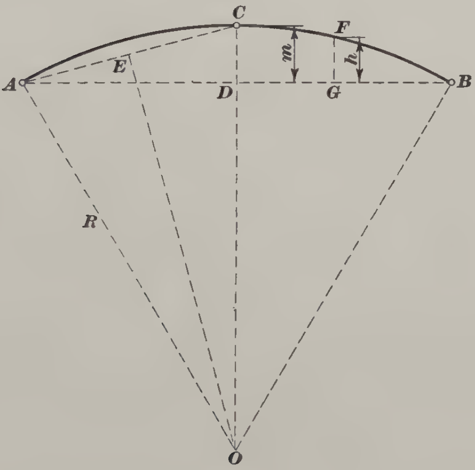

FIG. 20

Also, for determining the length of an ordinate, as FG, at any point on the curve, it is sufficiently accurate to apply the formula

$$h=\frac{b_1\times b_2}{2R} \qquad \textbf{(2) Approx.}$$

in which h=ordinate at any point, in feet;

b_1=distance from either extremity of chord to foot of ordinate, in feet;

b_2=distance from other extremity of chord to foot of ordinate, in feet;

R=radius of curve, in feet.

EXAMPLE.—A chord of 72 feet is laid off on a 6° curve, whose degree is based on a 100-foot chord. Compute (*a*) the middle ordinate, and (*b*) the ordinate 12 feet from the end of the chord.

SOLUTION.—(*a*) From Table I, $R=955.37$ ft.; and, by formula 1, the middle ordinate is

$$m=\frac{c^2}{8R}=\frac{72^2}{8\times 955.37}=0.678 \text{ ft.} \quad \text{Ans.}$$

(*b*) In formula 2, $b_1=12$ ft. and $b_2=72-12=60$ ft. Then the required ordinate is

$$h=\frac{b_1\times b_2}{2R}=\frac{12\times 60}{2\times 955.37}=0.377 \text{ ft.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. The P. C. of a 5° curve, whose degree is based on an arc of 100 feet, is at Sta. 29+49.4 and the total central angle is 17° 48′. Compute (*a*) the distance along the long chord from the P. C. to the foot of the perpendicular at Sta. 31; and (*b*) the ordinate from that chord to Sta. 31.

Ans. (*a*) 149.89 ft.
(*b*) 13.48 ft.

2. Compute the distance along the long chord from the P. T. to Sta. 32 in the preceding example. Ans. 104.74 ft.

3. A chord for a 14° curve, whose degree is based on a 100-foot chord, is 64 feet long. Determine the ordinate at a point on the chord 27 feet from one end. Ans. 1.22 ft.

OFFSETS FROM CHORDS PRODUCED

51. Where the degree of curve is based on a 100-foot chord and a transit is not available, it is sometimes convenient to locate the 100-foot stations on the curve by the method of offsets from chords produced, or the method of deflection distances. This method is employed mostly for restoring center-line stakes on a stretch of railroad track that has already been built. The general procedure is indicated in Fig. 21, where the tangents *AB* and *CD* are connected by the simple curve *BC*, and the stations *E*, *F*, *G*, and *H* are located by the method of offsets from chords produced.

The first station on the curve beyond the P. C. is located by the method of tangent offsets. Thus, the point *E* is located by setting the point *E′* on the tangent *AV* at the proper calculated distance from the P. C. and determining the intersection of the required tangent offset $E'E$ and the chord BE. The distances BE' and $E'E$ may be computed by formulas 1 and 2, Art. 44, or the offset $E'E$ may be calculated by applying formula 1 or 2, Art. 45.

Unless the P. C. is at a 100-foot station, in which case the chord BE is 100 feet long, it is customary to locate the second station in the following manner: The point B' is established at the intersection of two arcs. One is swung with E as a center and the distance BE' as the radius, and the other with B as a center and the distance $E'E$ as the radius. The line $B'E$ and its prolongation EF' will then be tangent to the curve at station E, and the point F may be located by the method of tangent offsets just as if E were the P. C. of a curve and F the first station on the curve. Since the chord EF is 100 feet long, the required distance EF'—from E to the foot of offset $F'F$ that is

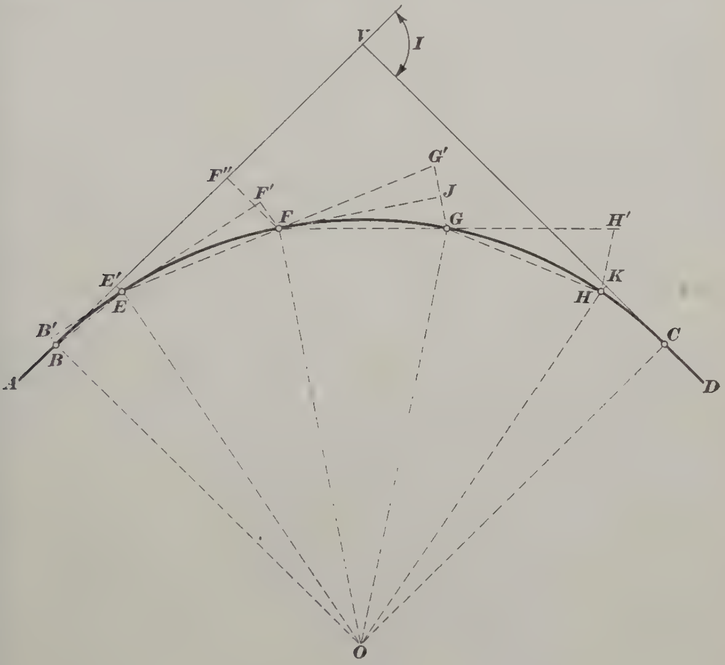

FIG. 21

perpendicular to the auxiliary tangent $B'F'$—is 100 cos $F'EF$ or 100 cos $\frac{1}{2}D$, where D denotes the degree of curve. Also, the offset $F'F$ is equal to 100 sin $\frac{1}{2}D$. Thus, the point F is established by marking the point F' and then locating the intersection of the offset $F'F$ and the chord EF.

Where BE is a short subchord, the auxiliary tangent $B'F'$ cannot be located accurately by the method just described. Consequently, in this case, both the first station at E and the second station at F should be located by offsets from the tangent at the P. C. Thus, F would be located by the distance BF'' along the tangent AV and the tangent offset $F''F$.

52. After the first two stations on the curve have been established, the remaining stations may be located by offsets from chords produced. Thus, the point G, Fig. 21, is located by prolonging the chord EF for a distance of 100 feet to G' and determining the intersection of the 100-foot chord FG and the offset $G'G$. The distance, such as GG', between a 100-foot chord and the prolongation of the preceding 100-foot chord is called the *chord deflection*. If the line FJ is drawn tangent to the curve at F, it will bisect the chord deflection $G'G$ at J and will also be perpendicular to $G'G$. Since the chords EF and FG are each 100 feet long, the triangles $EF'F$ and FJG are equal, JG is equal to $F'F$, and $G'G$ is twice $F'F$. In general, the chord deflection for any curve is equal to twice the tangent offset for a point that is 100 feet from the end of the curve. Values of the chord deflection for various degrees of curvature based on 100-foot chords are given in Table I in the column headed Chord Deflection.

The last station H on the curve shown in Fig. 21 may be located by prolonging the chord FG for 100 feet to H' and establishing the intersection of the chord GH and the chord deflection $H'H$. The position of point H may then be checked by calculating the coordinates CK and KH, along and perpendicular to the tangent CV at the P. T., and comparing them with the corresponding distances actually measured on the ground.

In case the P. C. is at a 100-foot station, the procedure is similar to that just described. However, the point E, Fig. 21, is then the P. C. and therefore it is unnecessary to locate an auxiliary tangent.

EXAMPLE.—The P. C. of a 4° curve, whose degree is based on a 100-foot chord, is at Sta. 8+25. Explain how to locate Stas. 9, 10, and 11 by the method of offsets from chords produced.

SOLUTION.—The conditions may be represented as in Fig. 21. In this problem the point B is at Sta. 8+25, the point E is at Sta. 9, and the length of the chord BE is 75 ft. Then, the central angle BOE is $D_s = 4 \times 0.75 = 3°$ and, by formulas 1 and 2, Art. 44, the distances BE' and $E'E$ are

$$BE' = 75 \cos 1° \, 30' = 74.97 \text{ ft.}$$

and

$$E'E = 75 \sin 1° \, 30' = 1.96 \text{ ft.}$$

Hence, Sta. 9 is located by first marking a point on the tangent through the P. C. at a distance of 74.97 ft. from the P. C. and then establishing the required station at a distance of 1.96 ft. from the point just set on the tangent and at a distance of 75 ft. from the P. C.

In order to locate Sta. 10, it is first necessary to fix the direction of the auxiliary tangent at Sta. 9. For this purpose, the point, as B' in Fig. 21, is set at a distance of 1.96 ft. from the P. C. and 74.97 ft. from Sta. 9. Also, the distance, as EF', along the tangent at Sta. 9 from that station to the foot of the offset to Sta. 10 is $100 \cos \frac{1}{2}D = 100 \cos 2° = 99.94$ ft. Hence, the tangent, as $B'E$, through Sta. 9 is prolonged beyond that station for a distance of 99.94 ft. and a stake is set at the end of the measurement. From Table I, the tangent offset for a 100-ft. chord of a 4° curve is found to be 3.49 ft., and Sta. 10 is located at that distance from the stake on the auxiliary tangent through Sta. 9 and at a distance of 100 ft. from Sta. 9, or at the intersection of the distances $F'F$ and EF in the illustration.

The first step in locating Sta. 11 is to prolong the chord between Stas. 9 and 10 for a distance of 100 ft. beyond Sta. 10; in other words, to set the point as G' in Fig. 21. Then, since from Table I the chord deflection for this curve is 6.98 ft., Sta. 11 is located at a distance of 6.98 ft. from the point, as G', on the prolongation of the chord through Stas. 9 and 10, and at a distance of 100 ft. from Sta. 10, or at the intersection of the distances $G'G$ and FG.

MIDDLE ORDINATES

53. The method of middle ordinates, which is another method for restoring the center-line stakes on a curve without a transit, is sometimes preferred to the method by offsets from chords produced. The usual procedure is illustrated in Fig. 22, where AB is the tangent at the P. C. of a curve and C, D, E, and F are the first four 100-foot stations on the curve. The first two stations, C and D, are located in the manner described in Art. 51 for the method of offsets from chords produced. Thus, the point C is established by the distances BC' and $C'C$ along and perpendicular to the tangent through the P. C. and by the length of the chord BC; and the point D is set by locating the auxiliary tangent $B'D'$ and measuring the distances CD', $D'D$, and CD.

To locate station E, the direction of the chord from C to E is first determined by setting the point G at a distance from station D equal to the middle ordinate for the chord CE. The middle ordinate DG should be laid off as nearly perpendicular to the required chord CE as can be estimated by eye. Since the distance between C and E along the curve is 200 feet, the length of the middle ordinate DG may be determined by substituting the degree of curve for $\frac{1}{2}I$ in formula 1 or 2 of Art. 49.

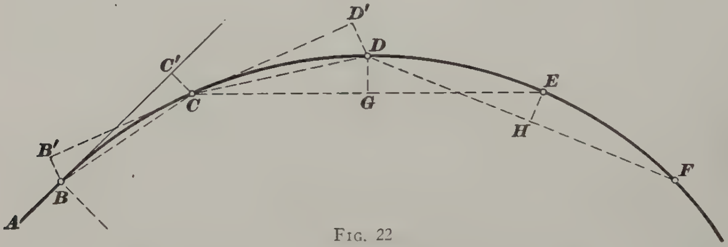

FIG. 22

However, this ordinate is equal to the tangent offset for a station 100 feet from the point of tangency, and thus is equal to the offset $D'D$, which has been previously used. The point E is established in line with points C and G and at a distance of 100 feet from the preceding station D.

Station F is located, in the manner just described for station E, by first laying off the middle ordinate EH and marking the point H on the ground; and then establishing the required station in line with D and H and at a distance of 100 feet from E.

SPECIAL FEATURES IN LAYING OUT SIMPLE CURVES

54. Selection of Degree of Curve.—The degree of the curve that is finally selected for connecting two given tangents usually depends on the character and importance of the route and on the topography of the ground surface. On important railroads and highways, the requirements of high-speed traffic impose a comparatively low limit on the maximum permissible degree of curve, and greater economy and safety in operating trains or motor vehicles are obtained by the use of flat curves. On the other hand, the cost of construction must not be overlooked, and in mountainous regions it may be necessary to use sharp curves in order to avoid excessive excavation or fill required to obtain suitable grades. Also, the smaller the degree of curve, the greater will be the length of curve necessary to effect a given change in direction. Consequently, the degree of curve is often limited by the available tangent distance.

Sometimes, special features govern the choice of the degree of curve. For example, the minimum or maximum distance from the vertex at the intersection of the tangents to the nearest point—which is the middle point—on the curve may be fixed by some feature of the topography or by a permanent obstruction. The straight-line distance from the vertex to the middle point of the curve is called the external distance and is usually designated by the letter E. Thus, in Fig. 19, the external distance is VQ. The external distance for any curve is part of the straight line, as VO, from the vertex to the center of the curve.

It is obvious from the foregoing discussion that the choice of the curve for a particular case requires careful study of the local conditions and is not based on any general rule.

55. Calculation of Radius From External Distance.—The external distance is not generally used in locating points on a curve, but it is often important in the selection of the degree of curve. When the intersection angle and the external distance for the curve are known, the radius can be readily computed. In Fig. 23, the P. C. is at A and the P. T. is at B, and the external distance VC is equal to $OV-OC$, or $OV=OC+VC$ $=R+E$, where R denotes the required radius of the curve and E the external distance. If I represents the intersection angle, it is evident that $OV=\frac{OA}{\cos AOV}=\frac{R}{\cos \frac{1}{2}I}$. Hence, $\frac{R}{\cos \frac{1}{2}I}=R+E$, from which $R=R \cos \frac{1}{2}I+E \cos \frac{1}{2}I$ and

$$R=\frac{E \cos \frac{1}{2}I}{1-\cos \frac{1}{2}I} \qquad (1)$$

This formula may also be written as

$$R=\frac{E}{\text{exsec } \frac{1}{2}I} \qquad (2)$$

The abbreviation *exsec* stands for exsecant, which is the trigonometric function of an angle that expresses the difference between the secant of the angle and 1. Thus, $\text{exsec } \frac{1}{2}I=\sec \frac{1}{2}I-1 =\frac{1}{\cos \frac{1}{2}I}-1=\frac{1-\cos \frac{1}{2}I}{\cos \frac{1}{2}I}$, and $\frac{\cos \frac{1}{2}I}{1-\cos \frac{1}{2}I}=\frac{1}{\text{exsec } \frac{1}{2}I}$.

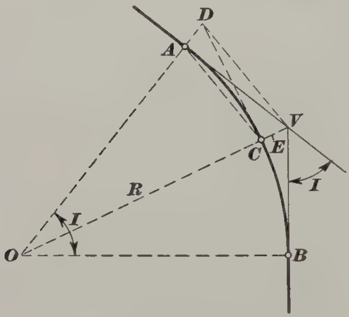

FIG. 23

When tables of secants or exsecants are not available and logarithms are used, the radius may be found conveniently by another relation among the external distance, the intersection angle, and the radius. This relation may be derived as follows: In Fig. 23, the line CD is drawn so that it is tangent to the curve at the middle point C and intersects the prolongation of the radius OA at D. Then, in the right triangle OCD, $CD=OC \tan DOC =R \tan \frac{1}{2}I$. Since this is also the value of the tangent distance for the curve AB, the distances CD and AV are equal. It now follows that the triangles ACD and ACV are equal, because each of the angles ACD and CAV is measured by one-half of the arc AC, and the side AC is common to the triangles; two sides and the included angle of one triangle are thus equal to two sides and the included angle of the other triangle. Therefore, $CV=AD$. Also, the line DV is parallel to the chord AC, and angle DVA is equal to angle VAC, which in turn is equal to $\frac{1}{2}AOC$ or $\frac{1}{4}I$. Hence, in the right triangle VAD, $AV=AD \cot DVA$, or $R \tan \frac{1}{2}I=E \cot \frac{1}{4}I$ and

$$R=E \cot \tfrac{1}{2}I \cot \tfrac{1}{4}I \qquad (3)$$

EXAMPLE.—Two tangents that intersect at an angle of 26° 40′ are to be connected by a curve whose center line at any point must not be less than 63 feet from the vertex. If the degree of curve is based on an arc of 100 feet, determine the required degree of curve.

SOLUTION.—In this case, $E = 63$ ft., $\frac{1}{2}I = 13° 20'$, and $\frac{1}{4}I = 6° 40'$. Then, the required radius is approximately equal to the value computed by formula 3, which is

$$R = E \cot \tfrac{1}{2}I \cot \tfrac{1}{4}I = 63 \cot 13° 20' \cot 6° 40' = 2{,}274 \text{ ft.}$$

From Table II, the next larger radius is 2,291.83 ft. This corresponds to a 2° 30′ curve, which will be used. Ans.

56. Checking Curve at Mid-Point.—Where a comparatively long curve is being located by deflection angles and the external distance from the vertex to the mid-point of the curve can be easily measured on the ground, it may be advisable to check the curve at the mid-point instead of waiting until the P. T. is reached. The procedure is then as follows: The external distance is first computed from the relation

$$E = R \operatorname{exsec} \tfrac{1}{2}I \qquad (1)$$

or

$$E = R \tan \tfrac{1}{2}I \tan \tfrac{1}{4}I \qquad (2)$$

in which the letters have the same meanings as in Art. 55. While the transit is set up at the vertex, a stake is placed at the mid-point of the curve, as C in Fig. 23, by backsighting along either tangent with the vernier at zero, turning off an angle equal to $\frac{1}{2}(180° - I)$ so that the line of sight will be directed toward the center of the curve, and laying off the external distance VC, or E, along this line.

When the mid-point is reached during the process of running in the curve by deflection angles, the position of that point, as determined by its deflection angle from the tangent through the P. C. and its distance from the preceding 100-foot station on the curve, should be reasonably close to the position previously established. The deflection angle to the mid-point is obviously $\frac{1}{4}I$ and the station number of that point is equal to the sum of the station of the P. C. and half the length of the curve.

57. Passing Obstacles on Curves.—There are numerous field conditions that may prevent the complete location of a curve by any single method, and the variety of problems encountered makes it impossible to give any general method for locating stations whose positions cannot be determined by the regular procedure. The best method of passing an obstacle on a curve will depend on the particular field conditions encountered and also on the ingenuity of the engineer in adapting the method to the conditions. Three of the more usual methods are indicated in Fig. 24.

In laying out the curve *ABCDEFGH* in view (*a*), stations *B* and *C* may be located by the method of deflection angles with the transit at the P. C. at *A*. However, between stations *C* and *D* there is a building that interferes with the measurement of the chord from *C* to *D* and with the line of sight from the P. C. to point *D*. Where the obstacle is not too far from one end of the curve, the method of tangent offsets would usually be given first consideration for locating station *D*. Thus, the point *D* could be located by measuring the distance *AD′* along the tangent at the P. C. and the offset *D′D* perpendicular to that tangent.

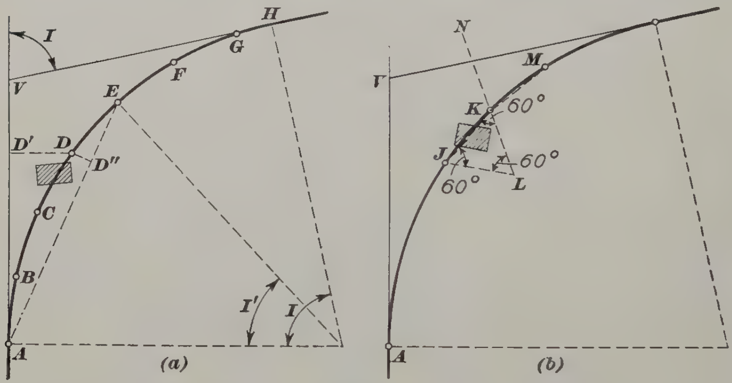

Fig. 24

In case the method of tangent offsets is not applicable, the next choice would be a modification of the method of ordinates from the long chord. In the illustration, the chord *AE* may be treated as the long chord for the curve, because the portion of the curve beyond *E* does not influence the position of *D* or *E*. Then the required distance *AD″* along this chord and the ordinate *D″D* at right angles to it may be calculated by applying the method of Art. 48 and using the central angle I' instead of the total angle I.

58. When an obstacle on a curve is several stations from either end of the curve and neither method described in the preceding article is convenient, the method illustrated in Fig. 24 (*b*) may be advantageously employed in passing the obstacle. After the curve has been laid out from the P. C. at A to station J near the obstacle, the station K on the other side of the obstacle may be located by running the lines JL and LK, each of which makes an angle of 60° with the chord JK and is given a length equal to the required length of that chord. Thus, the angle JLK must also be 60° and, if the field work in running the lines JL and LK is accurate, the point K will be in its proper position.

With the transit at J, the telescope can be brought into the line JL by a procedure similar to that described in Art. 34 or Art. 36 for determining the direction of the chord JK, but by setting the vernier to read the deflection angle to the point K plus 60°. The point L is next established at a distance from J equal to the required length of the chord JK, and the transit is moved to L. The direction of LK is then determined by turning an angle of 60° from LJ, and station K is located at a distance from L equal to the required length of JK.

In order to proceed along the curve from station K, the transit may be set up at that point and the line of sight directed along the chord KM in the following manner: A backsight is taken to L with the telescope inverted and the vernier reading 60° plus the deflection angle from a tangent at K to the chord KJ, or 60° plus one-half the degree of curve if JK is 100 feet long. The telescope is plunged back to normal so that the line of sight is along KN. Then the upper plate is unclamped, the telescope is turned past the zero reading of the vernier, and the vernier is set to read the deflection angle from the tangent at K to the chord KM, or one-half the degree of curve if KM is 100 feet. The remainder of the curve may then be run in by the method of deflection angles from the auxiliary tangent at K.

When a station point is inaccessible, another point on the curve near the inaccessible station may be located instead, but the station number of the point actually located must be recorded in the notes and marked on the stake driven at that point.

59. Locating Curve When P. I. is Inaccessible.—Wherever possible, the point of intersection of the tangents to a curve is located on the ground. However, the proposed route for a highway or railroad often winds around steep hills or runs along the bank of a river or the shore of a large lake. The point of intersection of two tangents that are to be connected by a curve may then be located either beyond the face of a hill or in deep water and, therefore, may be inaccessible. For instance, in Fig. 25, the intersection *V* of the tangents *AB* and *CD*, which are to be joined by the curve *BC*, is located in deep water. The method of procedure that is then adopted for establishing the P. C. and P. T. of the curve is usually as follows:

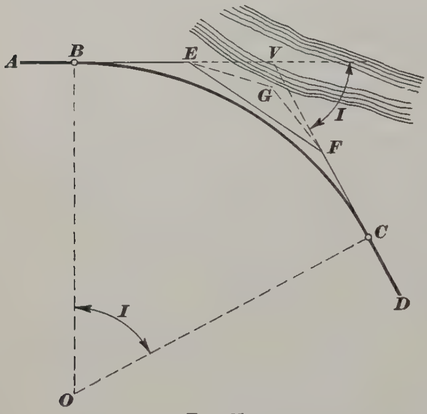

FIG. 25

From a point *E*, located at some convenient station on the extension of the tangent *AB*, either a line, as *EF*, or a traverse, as *EGF*, is run to a point *F* on the extension of the tangent *CD*. Where a straight line *EF* is used, the angles *VEF* and *VFE* and the distance *EF* are measured directly. In case a traverse is required, the angles *VEF* and *VFE* and the distance *EF* are computed from the traverse notes. The intersection angle *I* between the tangents *AB* and *CD* is an exterior angle of the triangle *VEF* and is equal to the sum of the angles *VEF* and *VFE*. Also, in the triangle *EFV*, the angle *EVF* is equal to 180° − I, and the lengths of the sides VE and VF can be computed by the relations

$$VE = \frac{EF \sin VFE}{\sin EVF}$$

$$VF = \frac{EF \sin VEF}{\sin EVF}$$

When the degree of the curve BC is selected, the tangent distance VB or VC may be computed by the formula of Art. 9, or $T = R \tan \frac{1}{2}I$. The required distance from the point E to the P. C. at B is then equal to $VB - VE$, and the distance from the point F to the P. T. at C is $VC - VF$.

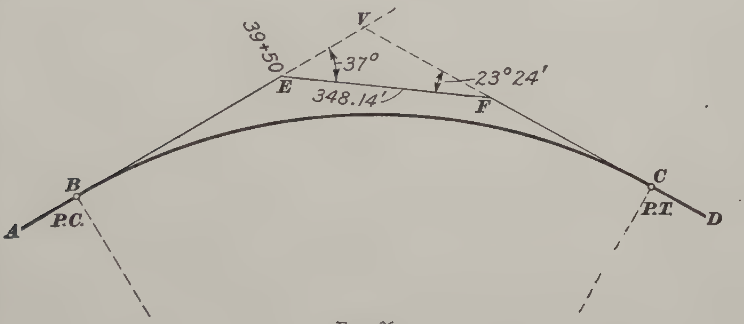

FIG. 26

EXAMPLE.—The intersection of two proposed tangents AB and CD, Fig. 26, is inaccessible. Therefore, from the point E, which is at Sta. 39+50 on the tangent AB, the line EF is run to the point F on the tangent CD. The angle between the prolongation of AB and the line EF, as measured by a transit, is found to be 37°, the length of EF is measured and found to be 348.14 feet, and the angle between EF and the prolongation of CD is 23° 24′. If the tangents AB and CD are to be connected by a 5° curve, whose degree is based on a chord of 100 feet, what should be (*a*) the station number of the P. C. on the tangent AB, and (*b*) the distance from F to the P. T.?

SOLUTION.—(*a*) Here, the angle of intersection between the tangents AB and CD is $I = VEF + VFE = 37° + 23° \, 24' = 60° \, 24'$ and angle $EVF = 180° - 60° \, 24' = 119° \, 36'$. Then, by solving the triangle EVF, it is found that

$$VE = \frac{EF \sin VFE}{\sin EVF} = \frac{348.14 \sin 23° \, 24'}{\sin 119° \, 36'} = 159.01 \text{ ft.}$$

and

$$VF = \frac{EF \sin VEF}{\sin EVF} = \frac{348.14 \sin 37°}{\sin 119° \, 36'} = 240.96 \text{ ft.}$$

The required tangent distance for the 5° curve is

$$VB = VC = R \tan \tfrac{1}{2} I = 1{,}146.28 \tan 30° \, 12' = 667.15 \text{ ft.}$$

Hence, the distance from E to the P. C. at B should be $VB - VE = 667.15 - 159.01 = 508.14$ ft. Obviously, the station number of the P. C. is less than that of point E. Since $3{,}950 - 508.14 = 3{,}441.86$, the P. C. should be at Sta. 34+41.86. Ans.

(*b*) The required distance from F to the P. T. is $VC - VF = 667.15 - 240.96 = 426.19$ ft. Ans.

60. To Replace Two Curves and a Tangent by a Single Curve.—In highway or railroad location it is usually desirable to use a single long curve rather than a combination of two shorter curves and a short tangent between them. Thus, in Fig. 27, the tangents *AB* and *CD* should preferably be connected by the single curve *BEC* rather than by the curves *BF* and *GH* and the short tangent *FG*. However, in running a preliminary line, it often happens that the three tangents *AJ*, *JK*, and *KD* are located in the field instead of the two tangents *AV* and *VD*. Where *JK* is a relatively short distance, it is disregarded in the final location survey, and a curve is run between the tangents *AV* and *VD*.

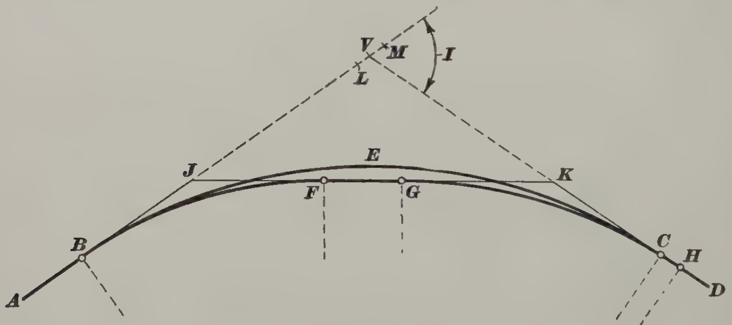

FIG. 27

If the point of intersection *V* of the tangents *AB* and *CD* is inaccessible, the P. C. at *B* can be located from the point *J*, and the P. T. at *C* can be located from the point *K*, as explained in the preceding article. If, on the other hand, it is possible to establish the point *V* on the ground, that point should be located, the angle *I* should be measured, and the P. C. and P. T.

at *B* and *C*, respectively, should be set by measuring the required tangent distance from *V*. In order to locate the point *V*, the following method may be used.

Range poles are set at two points on the tangent *CD*, such as points *K* and *D*. A transit is set up at *J*, a backsight is taken along the tangent line to *A*, with the telescope inverted, and the telescope is then plunged to its normal position. The flagman places a flag so that it is in the line of sight from the transit and also in line—as nearly as he can estimate—with the range poles at *K* and *D*. With a little practice, a flagman can determine closely the point of intersection of the two lines. The next step is to drive two temporary stakes, one on each side of the flag and along the prolongation of the tangent *AJ*, as at *L* and *M*. The flag is then removed, the direction of the line of sight is marked accurately by a tack driven half way in the top of each stake, and a cord is stretched between the two tacks.

The transit is now set up at *K*, a backsight is taken to *D* with the telescope inverted, and the telescope is plunged back to normal. A flag is then placed at the point where the line of sight of the transit intersects the cord stretched between the points *L* and *M*. This point is the intersection point *V* of the lines *AJ* and *KD* prolonged, or the P. I. of the proposed curve *BC*. A permanent stake is finally driven at *V*, and the P. I. is located accurately by driving a tack at the exact point where the prolongation of *KD*, as determined by the line of sight from the transit, crosses the cord connecting the tacks in the stakes at *L* and *M*.

61. Relocating Tangent.—After the preliminary survey and the subsequent calculations have been made, it is sometimes found that considerable excavation or fill can be avoided by moving the P. I. of a curve a short distance along one of the tangents and making the new tangent parallel to the old one. The intersection angle then remains unchanged, and the usual problem is either to retain the original degree of curve and to change the positions of the P. C. and P. T., or to retain the original P. C. or P. T. and to alter the degree of curve slightly.

62. A typical case in which a tangent is shifted is represented in Fig. 28. In the original location, the curve AB joins two tangents that meet at V. Later, it is found desirable to establish a new tangent $V'B'$ parallel to, and to the right of, the original position VB, and to insert a new curve $A'B'$ that has the same degree of curve as the old one. The distance AA' between the old and the new P. C. will then have a definite relation to the perpendicular distance BC between the tangent at the old P. T. and the new tangent. Since the intersection angle and the radius are the same for both the old and new curves, these two curves have the same tangent distance, or $AV = A'V'$ and $VB = V'B'$. Obviously, $AA' = VV'$; also, since VB is equal and parallel to $V'B'$, the figure $VV'B'B$ is a parallelogram. Hence, it follows that BB' is equal and parallel to VV', $AA' = BB'$, and angle $V'B'B = I$. If BC is drawn perpendicular to $V'B'$, then $BB' = \frac{BC}{\sin CB'B} = \frac{BC}{\sin I}$ or $AA' = \frac{BC}{\sin I}$. In general,

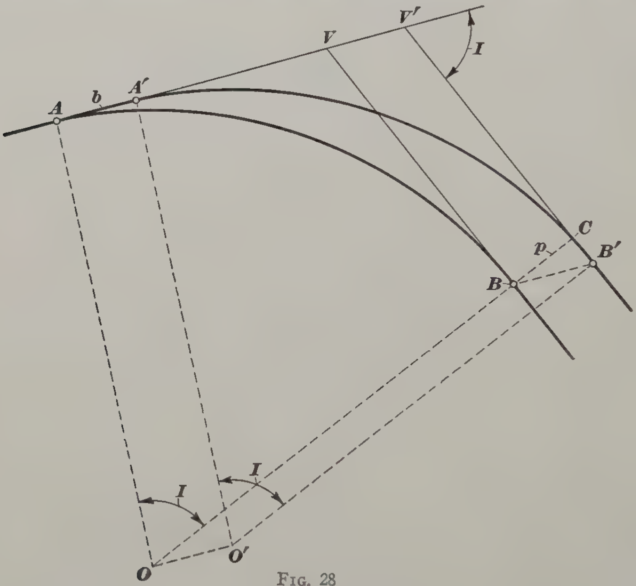

FIG. 28

$$b = \frac{p}{\sin I}$$

in which b = distance, in feet, between new and old P. C.;
p = perpendicular distance, in feet, between old tangent and new parallel tangent;
I = angle of intersection.

The new P. C. may be located on the tangent AV at A' by measuring the required distance AA' or b from the original P. C. at A. The new curve is then run in by any of the general methods for locating curves. In the illustration, the new tangent lies outside the old one, and the P. C. is moved toward the vertex. Where the new tangent is inside the old one, the same formula is applied for finding b, but the P. C. must be moved away from the vertex.

EXAMPLE.—A curve to the right, with its P. C. at Station 11+03.52, has a central angle of 27°. It has been found that the tangent through the P. T. of the curve should be moved outwards, or away from the center of the curve, a perpendicular distance of 21 feet. If the new curve is to be of the same degree as the old one, determine the station number of the new P. C.

SOLUTION.—In this case, $I = 27°$ and $p = 21$ ft.; then, the required distance from the old P. C. to the new one is

$$b = \frac{p}{\sin I} = \frac{21}{\sin 27°} = 46.26 \text{ ft.}$$

Since the new tangent is outside the old one, the P. C. is moved toward the vertex and its station number is increased. Thus, 1,103.52+46.26 = 1,149.78 and the new P. C. should be at Sta. 11+49.78. Ans.

63. In Fig. 29 are shown typical conditions where the P. C. of a curve is retained and, in order to meet a new tangent through the P. T., the degree of curve is altered. Here, AB is the original curve and BV is the tangent through its P. T. When the tangent through the P. T. is shifted parallel to itself to the position $V'B'$, the curve AB' is substituted for AB. Since the deflection angle from the tangent at the P. C. to the long chord of the curve is $\frac{1}{2}I$, regardless of the degree of curve, the long chord AB for the original curve must pass through the new P. T. at B'. If $B'C$ is drawn parallel to the radius OA at the P. C., then $B'C = OO'$, $OC = O'B'$, $BC = OB - O'B'$, and angle $B'CB = I$. Hence, if R is the radius OB of the original curve AB and R' is the radius $O'B'$ of the new curve AB', it follows that $B'C = R - R'$ and $BC = R - R'$. Also, the perpendicular distance between the new and old tangents is $p = BD = BC - CD = BC - B'C \cos B'CB = (R - R') - (R - R') \cos I = (R - R')(1 - \cos I)$; whence, $R - R' = \frac{p}{1 - \cos I}$ or, where the new tangent through the P. T. lies inside the old one,

$$R' = R - \frac{p}{1 - \cos I} \qquad (1)$$

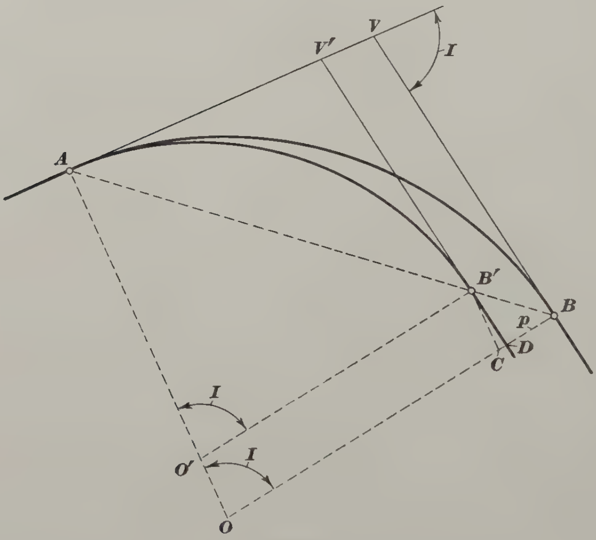

Fig. 29

in which R' = radius of required curve, in feet;
R = radius of original curve, in feet;
p = perpendicular distance between tangents, in feet;
I = intersection angle.

Where the new tangent through the P. T. is outside the old one, the required radius of the new curve is

$$R' = R + \frac{p}{1 - \cos I} \qquad (2)$$

64. Curve Joining Given Tangents and Passing Through Given Point.—Sometimes a railroad or highway curve, whose tangents are established, must pass through a certain point that is definitely located on the ground. The radius or degree of the curve is then fixed by this requirement. For example, in Fig. 30, the tangents AV and BV are to be joined by a curve that passes through the point C, and it is required to determine the radius of the curve.

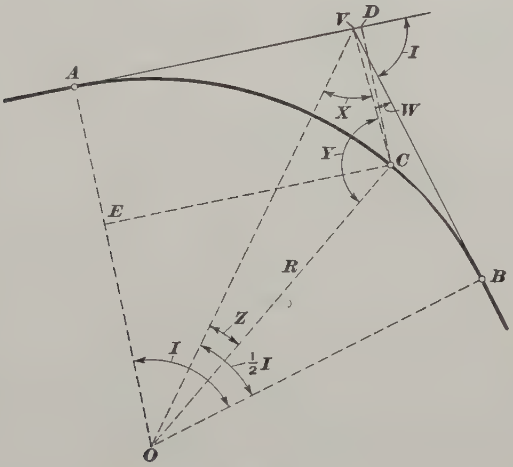

FIG. 30

The first step in solving this problem is to measure in the field the distance VC and the angle that the line VC makes with either tangent, as angle W in the illustration. Then, the required radius OC, or R, may be computed in the following manner: Since angle $VOB = \frac{1}{2}I$, angle $OVB = 90° - \frac{1}{2}I$, and $OVC = X = 90° - \frac{1}{2}I - W = 90° - (\frac{1}{2}I + W)$. Also, $OV = \frac{R}{\cos \frac{1}{2}I}$ and, by solving the triangle OVC, it is found that $\sin OCV = \frac{OV \sin OVC}{OC}$ or

$$\sin Y = \frac{\frac{R}{\cos \frac{1}{2}I} \sin X}{R} = \frac{\sin X}{\cos \frac{1}{2}I}$$

But, since $\sin X = \sin[90° - (\tfrac{1}{2}I + W)] = \cos(\tfrac{1}{2}I + W)$,

$$\sin Y = \frac{\cos(\frac{1}{2}I + W)}{\cos \frac{1}{2}I}$$

and the angle Y can thus be determined. In every case, this angle must be greater than 90° and the obtuse angle corresponding to the computed value of $\sin Y$ must always be used.

The radius can be calculated by the relation $OC = \dfrac{VC \sin OVC}{\sin VOC}$ or $R = \dfrac{VC \sin X}{\sin Z}$. However, as the angle Z is often very small, and a slight difference in Z would then make a great difference in $\sin Z$ and in R, it is usually preferable to determine R as follows: If the line CD is drawn perpendicular to the tangent AV produced, $CD = VC \sin DVC = VC \sin(I + W)$. Also, if CE is drawn perpendicular to the radius OA, $CD = OA - OE = OA - OC \cos EOC = R - R\cos(\tfrac{1}{2}I + Z)$. Hence, $R - R\cos(\tfrac{1}{2}I + Z) = VC \sin(I + W)$ and

$$R = \frac{VC \sin(I + W)}{1 - \cos(\frac{1}{2}I + Z)}$$

Example.—Two tangents, as AV and BV in Fig. 30, intersect at an angle of 54° 10′. The curve between these tangents must pass through a point that is located 162.4 feet from the P. I. and on a line from the vertex that makes an angle of 22° 27′ with the tangent to the P. T. Determine the radius of the required curve.

Solution.—In this case, $I = 54° 10'$ and $W = 22° 27'$. Hence,

$$\sin Y = \frac{\cos(\frac{1}{2}I + W)}{\cos \frac{1}{2}I} = \frac{\cos(27° 5' + 22° 27')}{\cos 27° 5'} = 0.72894$$

The acute angle whose sine is 0.72894 is 46° 48′ and $Y = 180° - 46° 48' = 133° 12'$. Since $X = 90° - (\tfrac{1}{2}I + W) = 90° - (27° 5' + 22° 27') = 40° 28'$, $Z = 180° - (X + Y) = 180° - (40° 28' + 133° 12') = 6° 20'$. For $VC = 162.4$ ft.,

$$R = \frac{VC \sin(I + W)}{1 - \cos(\frac{1}{2}I + Z)} = \frac{162.4 \sin(54° 10' + 22° 27')}{1 - \cos(27° 5' + 6° 20')} = 955.7 \text{ ft.} \quad \text{Ans.}$$

65. Intersection of Curve and Straight Line.—A problem that sometimes arises in practice is to locate the intersection of a curve with a certain straight line, such as a property line. Thus, in Fig. 31, the curve AB, whose tangents are AV and BV, intersects the straight line CD, and it is required to find the distance along the line CD from the tangent to the curve and also the station number on the curve of the point of intersection C. The first step is to measure on the ground the distance AD, from the nearer end of the curve to the point where the straight line intersects the tangent through that end; and also the angle ADC between the tangent and the intersecting line. Then, if R denotes the radius of the curve, $\tan ADO=\frac{R}{AD}$ and $OD=\frac{R}{\sin ADO}$. Also, angle $ODC=ADC-ADO$. In the triangle COD, the obtuse angle DCO can be determined by the relation

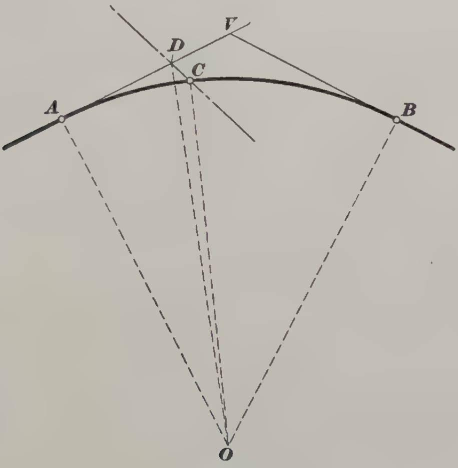

FIG. 31

$$\sin DCO=\frac{OD \sin ODC}{OC}$$

In the same triangle,

$$DOC=180°-(ODC+DCO)$$

and

$$DC=\frac{OC \sin DOC}{\sin ODC}$$

Also, angle $AOC=AOD+DOC$, and the required distance, in stations, along the curve from the P. C. to the point C where the curve intersects the straight line is found by dividing the value of the angle *AOC* by the degree of curve. The station number of the point of intersection can then be readily determined from the station number of the P. C.

EXAMPLE.—As indicated in Fig. 31, a 4° curve *AB*, whose degree is based on a 100-foot arc, and whose P. C. is at Sta. 19+61.3, intersects a property line *CD* at the point *C*. The tangent *A V* through the P. C. of the curve intersects the property line at point *D*, which is 489.6 feet from the P. C.; also, the angle *ADC* between the tangent and the property line is 111° 45′. Determine (*a*) the distance along the property line from the tangent to the curve, and (*b*) the station number on the curve of the point where the curve intersects the property line.

SOLUTION.—(*a*) The radius of the curve is 1,432.4 ft. and, since $AD = 489.6$ ft.,

$$\tan ADO = \frac{1{,}432.4}{489.6} \text{ and } ADO = 71^\circ\ 8'$$

Therefore, $OD = \dfrac{R}{\sin 71^\circ\ 8'} = 1{,}513.7$ ft. and angle $ODC = ADC - ADO = 111^\circ\ 45' - 71^\circ\ 8' = 40^\circ\ 37'$.

Also, $\sin DCO = \dfrac{OD \sin ODC}{OC} = \dfrac{1{,}513.7 \sin 40^\circ\ 37'}{1{,}432.4}$ and, since *DCO* is obviously an obtuse angle,

$$DCO = 180^\circ - 43^\circ\ 28' = 136^\circ\ 32'$$

Hence, $DOC = 180^\circ - (ODC + DCO) = 180^\circ - (40^\circ\ 37' + 136^\circ\ 32') = 2^\circ\ 51'$.

The required distance along the property line from the tangent to the curve is

$$DC = \frac{OC \sin DOC}{\sin ODC} = \frac{1{,}432.4 \sin 2^\circ\ 51'}{\sin 40^\circ\ 37'} = 109.4 \text{ ft.} \quad \text{Ans.}$$

(*b*) Angle $AOD = 90^\circ - ADO = 18^\circ\ 52'$ and the central angle *AOC*, between the radius to the P. C. and that to the intersection *C* of the curve and the property line, is $AOD + DOC = 18^\circ\ 52' + 2^\circ\ 51' = 21^\circ\ 43'$ or 21.717°. Hence, the distance along the curve from the P. C. to the intersection with the property line is $21.717 \div 4 = 5.429$ sta. Since $19.613 + 5.429 = 25.042$, the station number of point *C* is 25+04.2. Ans.

EXAMPLES FOR PRACTICE

1. The P. I. of two tangents that are to be connected by a curve is inaccessible. A straight line 97 feet long from Sta. 44 on the tangent through the P. C. to a point on the other tangent makes an angle of 42° with the tangent through the P. C. and an angle of 26° 42′ with the tangent through the P. T. If the tangents are joined by a 3° curve whose degree is based on a 100-foot chord, what is the station number of the P. C.?

Ans. 31+41.37

2. The angle of intersection of two tangents is 61° 30′. A curve joining these tangents is to pass through a point that is 129 feet from the P. I. and on a line from the vertex that makes an angle of 16° 29′ with the tangent through the P. C. Determine the radius of the required curve. Ans. 534.6 ft.

3. The P. C. of a curve to the left is at Station 9+16.27 and the central angle is 31° 45′. The tangent through the P. T. of the curve is to be moved away from the center a perpendicular distance of 16 feet. If the new curve is to be of the same degree as the old one, determine the station number of the new P. C. Ans. 9+46.68

DEGREE OF CURVE OF EXISTING TRACK

66. Outline of Usual Procedure.—It is sometimes desired to determine the degree of a curve in an existing railroad track. The simplest method is to measure the length of the chord between two points on the curve and the length of the middle ordinate from that chord to the curve. The required radius or degree of the curve can then be readily computed from these two measured values. Since the curvature of the track will seldom be exactly uniform for its entire length, it is necessary to find the middle ordinates for several chords of the same length at different parts of the curve, and the average of these middle ordinates is used in calculating the radius or degree of the curve. It is sufficiently accurate to take measurements only along the outside rail of the track.

67. Computation of Radius.—Any method of determining the radius or degree of an existing curve by measurements on the track will necessarily be approximate, because the track will not be in perfect alinement. Therefore, extreme refinement in the measurements and computations is not warranted, and the approximate relation expressed by formula 1, Art. 50, is a sufficiently accurate basis for computing the radius of the curve from the length of chord and the middle ordinate. From this formula,

$$R=\frac{c^2}{8m}$$

in which R=radius of curve, in feet:
c=length of chord, in feet;
m=middle ordinate, in feet.

68. Determination of Degree of Curve.—The degree of an existing curve may be computed from its radius by the relation $D=\frac{5{,}730}{R}$, whether the degree is based on an arc or a chord of 100 feet. Since the value of the radius that would be used is not accurate, the degree of curve cannot be determined exactly and the result is usually taken to the nearest practical value. For example, if the calculated value of the degree of curve is found to be 5.04°, it would be assumed that the original curve was a 5° curve. If the value $\frac{5{,}730}{D}$ is substituted for R in the formula of the preceding article, $\frac{5{,}730}{D}=\frac{c^2}{8m}$. Hence, the degree of curve D may be found directly from the chord c, in feet, and the middle ordinate m, in feet, by the formula

$$D=\frac{45{,}840m}{c^2} \qquad (1)$$

Often, it is convenient to measure the middle ordinate in inches, rather than in feet. If m' represents the ordinate in inches, $m=\frac{1}{12}m'$ and, from formula 1, $D=\frac{45{,}840\times\frac{1}{12}m'}{c^2}$, or

$$D=\frac{3{,}820m'}{c^2} \qquad (2)$$

in which D = degree of curve;
m' = middle ordinate, in inches;
c = length of chord, in feet.

69. Convenient Length of Chord.—In order to simplify the calculation of the degree of curve, it is customary to select a chord length that will make the degree of curve numerically equal to the middle ordinate. If the middle ordinate is measured in inches, the value of c^2 should be equal to 3,820, which will be the case when the length of chord equals $\sqrt{3{,}820}=61.8$ feet. In railroad practice, a length of 62 feet is commonly used. Hence, the degree of an existing curve is usually found by laying off several 62-foot chords in various parts of the curve, determining the middle ordinate for each such chord in inches and a decimal part of an inch, and averaging these ordinates.

TABLE I—RADII AND DEFLECTION DISTANCES

(For degree of curve defined as angle subtended by chord of 100 feet)

Degree	Radius	Chord Deflection	Tangent Offset
° ′			
0 5	68754.94	.145	.073
10	34377.48	.291	.145
15	22918.33	.436	.218
20	17188.76	.582	.291
25	13751.02	.727	.364
30	11459.19	.873	.436
35	9822.18	1.018	.509
40	8594.41	1.164	.582
45	7639.49	1.309	.654
50	6875.55	1.454	.727
55	6250.51	1.600	.800
1 0	5729.65	1.745	.873
5	5288.92	1.891	.945
10	4911.15	2.036	1.018
15	4583.75	2.182	1.091
20	4297.28	2.327	1.164
25	4044.51	2.472	1.236
30	3819.83	2.618	1.309
35	3618.80	2.763	1.382
40	3437.87	2.909	1.454
45	3274.17	3.054	1.527
50	3125.36	3.200	1.600
55	2989.48	3.345	1.673
2 0	2864.93	3.490	1.745
5	2750.35	3.636	1.818
10	2644.58	3.781	1.891
15	2546.64	3.927	1.963
20	2455.70	4.072	2.036
25	2371.04	4.218	2.109
30	2292.01	4.363	2.181
35	2218.09	4.508	2.254
40	2148.79	4.654	2.327
45	2083.68	4.799	2.400
50	2022.41	4.945	2.472
55	1964.64	5.090	2.545
3 0	1910.08	5.235	2.618
5	1858.47	5.381	2.690
10	1809.57	5.526	2.763
15	1763.18	5.672	2.836
20	1719.12	5.817	2.908
25	1677.20	5.962	2.981
30	1637.28	6.108	3.054
35	1599.21	6.253	3.127
40	1562.88	6.398	3.199
45	1528.16	6.544	3.272
50	1494.95	6.689	3.345
55	1463.16	6.835	3.417
4 0	1432.69	6.980	3.490
5	1403.46	7.125	3.563
10	1375.40	7.271	3.635
15	1348.45	7.416	3.708
20	1322.53	7.561	3.781
25	1297.58	7.707	3.853
30	1273.57	7.852	3.926
35	1250.42	7.997	3.999
40	1228.11	8.143	4.071
45	1206.57	8.288	4.144
50	1185.78	8.433	4.217
55	1165.70	8.579	4.289
5 0	1146.28	8.724	4.362
5	1127.50	8.869	4.435
10	1109.33	9.014	4.507
5 15	1091.73	9.160	4.580
20	1074.68	9.305	4.653
25	1058.16	9.450	4.725
30	1042.14	9.596	4.798
35	1026.60	9.741	4.870
40	1011.51	9.886	4.943
45	996.87	10.031	5.016
50	982.64	10.177	5.088
55	968.81	10.322	5.161
6 0	955.37	10.467	5.234
5	942.29	10.612	5.306
10	929.57	10.758	5.379
15	917.19	10.903	5.451
20	905.13	11.048	5.524
25	893.39	11.193	5.597
30	881.95	11.339	5.669
35	870.79	11.484	5.742
40	859.92	11.629	5.814
45	849.32	11.774	5.887
50	838.97	11.919	5.960
55	828.88	12.065	6.032
7 0	819.02	12.210	6.105
5	809.40	12.355	6.177
10	800.00	12.500	6.250
15	790.81	12.645	6.323
20	781.84	12.790	6.395
25	773.07	12.936	6.468
30	764.49	13.081	6.540
35	756.10	13.226	6.613
40	747.89	13.371	6.685
45	739.86	13.516	6.758
50	732.01	13.661	6.831
55	724.31	13.806	6.903
8 0	716.78	13.951	6.976
5	709.40	14.096	7.048
10	702.18	14.241	7.121
15	695.09	14.387	7.193
20	688.16	14.532	7.266
25	681.35	14.677	7.338
30	674.69	14.822	7.411
35	668.15	14.967	7.483
40	661.74	15.112	7.556
45	655.45	15.257	7.628
50	649.27	15.402	7.701
55	643.22	15.547	7.773
9 0	637.27	15.692	7.846
5	631.44	15.837	7.918
10	625.71	15.982	7.991
15	620.09	16.127	8.063
20	614.56	16.272	8.136
25	609.14	16.417	8.208
30	603.80	16.562	8.281
35	598.57	16.707	8.353
40	593.42	16.852	8.426
45	588.36	16.996	8.498
50	583.38	17.141	8.571
55	578.49	17.286	8.643
10 0	573.69	17.431	8.716
10	564.31	17.721	8.860
20	555.23	18.011	9.005
30	546.44	18.300	9.150
40	537.92	18.590	9.295
10 50	529.67	18.880	9.440
11 00	521.67	19.169	9.585
10	513.91	19.459	9.729
20	506.38	19.748	9.874
30	499.06	20.038	10.019
40	491.96	20.327	10.164
50	485.05	20.616	10.308
12 0	478.34	20.906	10.453
10	471.81	21.195	10.597
20	465.46	21.484	10.742
30	459.28	21.773	10.887
40	453.26	22.063	11.031
50	447.40	22.352	11.176
13 0	441.68	22.641	11.320
10	436.12	22.930	11.465
20	430.69	23.219	11.609
30	425.40	23.507	11.754
40	420.23	23.796	11.898
50	415.19	24.085	12.043
14 0	410.28	24.374	12.187
10	405.47	24.663	12.331
20	400.78	24.951	12.476
30	396.20	25.240	12.620
40	391.72	25.528	12.764
50	387.34	25.817	12.908
15 0	383.06	26.105	13.053
10	378.88	26.394	13.197
20	374.79	26.682	13.341
30	370.78	26.970	13.485
40	366.86	27.258	13.629
50	363.02	27.547	13.773
16 0	359.26	27.835	13.917
10	355.59	28.123	14.061
20	351.98	28.411	14.205
30	348.45	28.699	14.349
40	344.99	28.986	14.493
50	341.60	29.274	14.637
17 0	338.27	29.562	14.781
10	335.01	29.850	14.925
20	331.82	30.137	15.069
30	328.68	30.425	15.212
40	325.60	30.712	15.356
50	322.59	31.000	15.500
18 0	319.62	31.287	15.643
10	316.71	31.574	15.787
20	313.86	31.861	15.931
30	311.06	32.149	16.074
40	308.30	32.436	16.218
50	305.60	32.723	16.361
19 0	302.94	33.010	16.505
10	300.33	33.296	16.648
20	297.77	33.583	16.792
30	295.25	33.870	16.935
40	292.77	34.157	17.078
50	290.33	34.443	17.222
20 0	287.94	34.730	17.365

TABLE II—RADII AND CHORD LENGTHS

(For degree of curve defined as angle subtended by arc of 100 feet)

Degree	Radius	Chord for 100 ft. of Arc	Degree	Radius	Chord for 50 ft. of Arc	Chord for 100 ft. of Arc	Degree	Radius	Chord for 25 ft. of Arc	Chord for 50 ft. of Arc	Chord for 100 ft. of Arc
° ′			° ′				° ′				
0 5	68,754.94	100.00	5 15	1,091.35	50.00	99.97	10 50	528.88		49.98	99.85
10	34,377.47	100.00	20	1,074.30	50.00	99.96					
15	22,918.31	100.00	25	1,057.77	50.00	99.96	11 0	520.87		49.98	99.85
20	17,188.73	100.00	30	1,041.74	50.00	99.96	10	513.10		49.98	99.84
25	13,750.99	100.00	35	1,026.19	50.00	99.96	20	505.55		49.98	99.84
30	11,459.16	100.00	40	1,011.10	49.99	99.96	30	498.22		49.98	99.83
35	9,822.13	100.00	45	996.45	49.99	99.96	40	491.11		49.98	99.83
40	8,594.37	100.00	50	982.21	49.99	99.96	50	484.19		49.98	99.82
45	7,639.44	100.00	55	968.38	49.99	99.96					
50	6,875.49	100.00					12 0	477.47		49.98	99.82
55	6,250.45	100.00	6 0	954.93	49.99	99.95	10	470.92		49.98	99.81
			5	941.85	49.99	99.95	20	464.56		49.98	99.81
1 0	5,729.58	100.00	10	929.12	49.99	99.95	30	458.37		49.98	99.80
5	5,288.84	100.00	15	916.73	49.99	99.95	40	452.34		49.97	99.80
10	4,911.07	100.00	20	904.67	49.99	99.95	50	446.46		49.97	99.79
15	4,583.66	100.00	25	892.92	49.99	99.95					
20	4,297.18	100.00	30	881.47	49.99	99.95	13 0	440.74		49.97	99 79
25	4,044.41	100.00	35	870.32	49.99	99.94	10	435.16		49.97	99.78
30	3,819.72	100.00	40	859.44	49.99	99.94	20	429.72		49.97	99.77
35	3,618.68	100.00	45	848.83	49.99	99.94	30	424.41		49.97	99.77
40	3,437.75	100.00	50	838.47	49.99	99.94	40	419.24		49.97	99.76
45	3,274.04	100.00	55	828.37	49.99	99.94	50	414.19		49.97	99.76
50	3,125.22	100.00									
55	2,989.35	100.00	7 0	818.51	49.99	99.94	14 0	409.26		49.97	99.75
			5	808.88	49.99	99.94	10	404.44		49.97	99.75
2 0	2,864.79	99.99	10	799.48	49.99	99.93	20	399.74		49.97	99.74
5	2,750.20	99.99	15	790.29	49.99	99.93	30	395.14		49.97	99.73
10	2,644.42	99.99	20	781.31	49.99	99.93	40	390.65		49.97	99.73
15	2,546.48	99.99	25	772.53	49.99	99.93	50	386.26		49.97	99.72
20	2,455.53	99.99	30	763.94	49.99	99.93					
25	2,370.86	99.99	35	755.55	49.99	99.93	15 0	381.97	25.00	49.96	99.72
30	2,291.83	99.99	40	747.34	49.99	99.92	10	377.77	25.00	49.96	99.71
35	2,217.90	99.99	45	739.30	49.99	99.92	20	373.67	25.00	49.96	99.70
40	2,148.59	99.99	50	731.44	49.99	99.92	30	369.65	25.00	49.96	99.70
45	2,083.48	99.99	55	723.74	49.99	99.92	40	365.72	25.00	49.96	99.69
50	2,022.20	99.99					50	361.87	25.00	49.96	99.68
55	1,964.43	99.99	8 0	716.20	49.99	99.92					
			5	708.81	49.99	99.92	16 0	358.10	24.99	49.96	99.68
3 0	1,909.86	99.99	10	701.58	49.99	99.92	10	354.41	24.99	49.96	99.67
5	1,858.24	99.99	15	694.49	49.99	99.91	20	350.79	24 99	49.96	99.66
10	1,809.34	99.99	20	687.55	49.99	99.91	30	347.25	24.99	49.96	99.65
15	1,762.95	99.99	25	680.74	49.99	99.91	40	343.77	24.99	49.96	99.65
20	1,718.87	99.99	30	674.07	49.99	99.91	50	340.37	24.99	49.96	99.64
25	1,676.95	99.99	35	667.52	49.99	99.91					
30	1,637.02	99.98	40	661.11	49.99	99.90	17 0	337.03	24.99	49.95	99.63
35	1,598.95	99.98	45	654.81	49.99	99.90	10	333.76	24.99	49.95	99.63
40	1,562.61	99.98	50	648.63	49.99	99.90	20	330.55	24.99	49.95	99.62
45	1,527.89	99.98	55	642.57	49.99	99.90	30	327.40	24.99	49.95	99.61
50	1,494.67	99.98					40	324.32	24.99	49.95	99.60
55	1,462.87	99.98	9 0	636.62	49.99	99.90	50	321.28	24.99	49.95	99.60
			5	630.78	49.99	99.89					
4 0	1,432.40	99.98	10	625.04	49.99	99.89	18 0	318.31	24.99	49.95	99.59
5	1,403.16	99.98	15	619.41	49.99	99.89	10	315.39	24.99	49.95	99.58
10	1,375.10	99.98	20	613.88	49.99	99.89	20	312.52	24.99	49.95	99.57
15	1,348.14	99.98	25	608.45	49.99	99.89	30	309.71	24.99	49.95	99.57
20	1,322.21	99.98	30	603.11	49.99	99.89	40	306.94	24.99	49.94	99.56
25	1,297.26	99.98	35	597.87	49.99	99.88	50	304.23	24.99	49.94	99.55
30	1,273.24	99.97	40	592.72	49.99	99.88					
35	1,250.09	99.97	45	587.65	49.99	99.88	19 0	301.56	24.99	49.94	99.54
40	1,227.77	99.97	50	582.67	49.98	99.88	10	298.93	24.99	49.94	99.53
45	1,206.23	99.97	55	577.77	49.98	99.88	20	296.36	24.99	49.94	99.53
50	1,185.43	99.97					30	293.82	24.99	49.94	99.52
55	1,165.34	99.97	10 0	572.96	49.98	99.87	40	291.34	24.99	49.94	99.51
			10	563.57	49.98	99.87	50	288.89	24.99	49.94	99.50
5 0	1,145.92	99.97	20	554.48	49.98	99.86					
5	1,127.13	99.97	30	545.67	49.98	99.86	20 0	286.48	24.99	49.94	99.49
10	1,108.95	99.97	40	537.15	49.98	99.86					

CIRCULAR AND PARABOLIC CURVES

Serial 2912B-2

(PART 2)

Edition 1

COMPOUND AND REVERSE CURVES

COMPOUND CURVES

INTRODUCTION

1. Use of Compound Curves.—Sometimes the ground condition is such that no simple circular curve is well adapted to the topography. In mountainous country, where the road may wind along a valley or up a mountainside, it is often necessary to use compound curves in order to locate the road in the most favorable position.

2. Required Quantities for Compound Curves.—In Fig. 1 is shown a compound curve ABC, which consists of two simple circular curves AB and BC, whose radii are OA and $O'B$, respectively. The lines AD and CD are tangent to the compound curve at its extremities, and the line EF is tangent to the curve at the point of compound curvature, which is usually abbreviated P.C.C. In laying out the compound curve ABC in the field, the following seven quantities are important: (1) The angle of intersection I between the tangents AD and CD; (2) the angle I_1 between the tangents AD and EF, which is equal to the central angle of the curve AB; (3) the angle I_2 between the tangents CD and EF, which is equal to the central angle of the curve BC; (4) the degree of the flatter curve AB; (5) the degree of the sharper curve BC; (6) the length of the tangent AD; and (7) the length of the tangent CD. Of course, the radius of a curve can be readily found from the degree, and conversely.

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When any four of the seven quantities here mentioned are given, the other three can be found by applying the principles of geometry and trigonometry. However, instead of working out each problem entirely as an individual case, it is convenient to use general formulas that apply to the particular conditions.

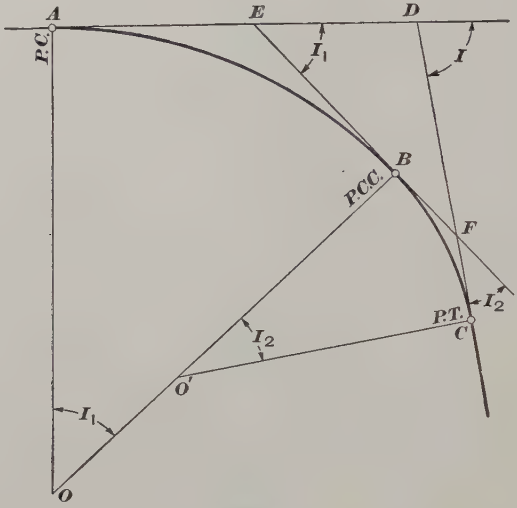


FIG. 1

COMPUTATIONS FOR COMPOUND CURVES

3. Fundamental Formulas.—For the purpose of solving the various problems that may arise in calculating the required quantities for a compound curve, there are seven fundamental formulas. In Fig. 2, *ABC* is a compound curve joining the tangents *AD* and *CD* whose angle of intersection is I. Also, *EF* is the tangent at the P. C. C., the angle of intersection between *EF* and *AD* is I_1, and that between *EF* and *CD* is I_2. The radius *OA* or *OB* of the flatter curve *AB* is R_1, and the radius *O′B* or *O′C* of the sharper curve *BC* is R_2. The tangent distance *AD* is denoted by T_1, and the tangent distance *CD* by T_2. Since the angle *GDF* is an exterior angle of the triangle *DEF*, it is equal to the sum of the opposite interior angles *DEF* and *DFE*, or

$$I = I_1 + I_2 \qquad (1)$$

In order to derive other formulas, the line OH is drawn parallel to $O'C$, and the line $O'J$ parallel to OA; the curve AB is prolonged to H, and the curve BC is prolonged to K; and with O as a center and OO' as a radius, arc LM is drawn. Also, CN is drawn perpendicular to $O'J$, and $O'P$ perpendicular to OA. Then, $MP=AP-AM$ and $JK=O'J-O'K$. But AP and $O'J$ are equal, and AM is equal to $O'K$ because each is equal to $O'B$. Therefore, $MP=JK$. The next step is to express the distances MP and JK in terms of other quantities and to equate the values so obtained.

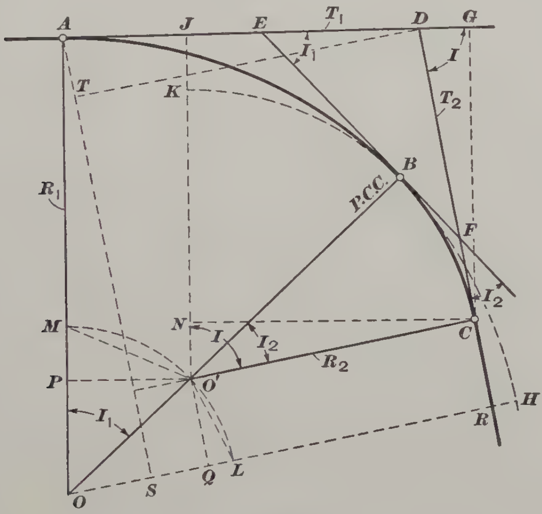


FIG. 2

Thus, $MP=OM-OP$, $OM=OO'=OB-O'B=R_1-R_2$, and $OP=OO'\cos I_1=(R_1-R_2)\cos I_1$. Hence, an expression for MP is as follows:

$$MP=(R_1-R_2)-(R_1-R_2)\cos I_1=(R_1-R_2)(1-\cos I_1)$$

Also, $JK=JN-KN$, $JN=CG=CD\sin I=T_2\sin I$, and $KN=O'K-O'N=R_2-R_2\cos I=R_2(1-\cos I)$. Therefore,

$$JK=T_2\sin I-R_2(1-\cos I)$$

Then, since $MP=JK$,

$$(R_1-R_2)(1-\cos I_1)=T_2\sin I-R_2(1-\cos I) \qquad (2)$$

In the illustration, $O'P = AG - CN = AD + DG - CN$. But $O'P = OO' \sin I_1 = (R_1 - R_2) \sin I_1$, $AD = T_1$, $DG = T_2 \cos I$, and $CN = O'C \sin I = R_2 \sin I$. Hence,

$$(R_1 - R_2) \sin I_1 = T_1 + T_2 \cos I - R_2 \sin I \qquad (3)$$

To derive the next formula, the chord $O'M$ is drawn. Since $O'P$ is perpendicular to OM, the angle $MO'P$ between the chord $O'M$ and the half-chord $O'P$ is equal to half of the angle I_1. Also, $\tan MO'P = \frac{MP}{O'P}$. Hence, if the values previously determined are substituted in this relation,

$$\tan \tfrac{1}{2} I_1 = \frac{T_2 \sin I - R_2(1 - \cos I)}{T_1 + T_2 \cos I - R_2 \sin I} \qquad (4)$$

The remaining formulas are derived by methods similar to those explained for formulas 2, 3, and 4. Thus, $O'Q$ is drawn perpendicular to OH; then, as $LQ = QR - LR$, $QR = O'C = O'B = LH$, and $HR = LH - LR$, it follows that $LQ = HR$. Expressions for LQ and HR are now developed and equated, as follows: $LQ = OL - OQ = (R_1 - R_2)(1 - \cos I_2)$, the angle LOO' being equal to $BO'C$ or I_2. Also, if AS is drawn perpendicular to OH, and DT parallel to OH, then $HR = HS - DT$. But, $HS = OH - OS = R_1(1 - \cos I)$, the angle AOS being equal to GDC or I; and $DT = AD \sin DAT = T_1 \sin I$, the angle DAT also being equal to I. Hence,

$$(R_1 - R_2)(1 - \cos I_2) = R_1(1 - \cos I) - T_1 \sin I \qquad (5)$$

Also, in the illustration, $O'Q = AS - AT - CD$. Since $O'Q = OO' \sin O'OQ = (R_1 - R_2) \sin I_2$, $AS = OA \sin AOS = R_1 \sin I$, $AT = AD \cos DAT = T_1 \cos I$, and $CD = T_2$,

$$(R_1 - R_2) \sin I_2 = R_1 \sin I - T_1 \cos I - T_2 \qquad (6)$$

If the chord $O'L$ is drawn, angle $LO'Q = \tfrac{1}{2} O'OL = \tfrac{1}{2} I_2$ and $\tan LO'Q = \frac{LQ}{O'Q}$, or

$$\tan \tfrac{1}{2} I_2 = \frac{R_1(1 - \cos I) - T_1 \sin I}{R_1 \sin I - T_1 \cos I - T_2} \qquad (7)$$

For convenience, the meanings of the letters in the preceding formulas are summarized as follows:

I = angle of intersection between tangents to compound curve at its extremities;
R_1 = radius of flatter simple curve;
R_2 = radius of sharper simple curve;
I_1 = central angle of simple curve whose radius is R_1;
I_2 = central angle of simple curve whose radius is R_2;
T_1 = length of tangent at extremity of flatter curve;
T_2 = length of tangent at extremity of sharper curve.

4. **Solutions of Typical Problems.**—In the usual problems on compound curves, three of the given quantities are the angle of intersection between the tangents at the extremities of the curve, the degree or radius of one simple curve, and the length of one tangent. The fourth known value may be the degree or radius of the other curve, the other tangent distance, or the central angle of either simple curve. The formulas to be used in any problem, and the order in which they are applied, depend on what quantities are given.

The methods of solving the usual problems on compound curves are illustrated by the following typical cases. In the numerical examples given for the various cases, it is assumed that the degree of curve is based on a 100-foot chord.

5. **Case I.**—Given I, R_1, R_2, and T_2; to find I_1, I_2, and T_1. In this case, I_1 can be found first by applying formula 2, Art. 3, which becomes

$$1-\cos I_1=\frac{T_2 \sin I-R_2(1-\cos I)}{R_1-R_2} \qquad (1)$$

Next, I_2 is determined from formula 1, which may be written as

$$I_2=I-I_1 \qquad (2)$$

Finally, T_1 is calculated by the following formula, which is derived from formula 3, Art. 3:

$$T_1=(R_1-R_2)\sin I_1-T_2\cos I+R_2\sin I \qquad (3)$$

EXAMPLE.—Two tangents intersecting at an angle of 35° 20′ are to be joined by a compound curve consisting of a 3° curve and a 5° curve. If the tangent to the compound curve at the extremity of the 5° curve is to be 404 feet long, determine (*a*) the central angle of each curve and (*b*) the length of the other tangent.

Solution.—(*a*) Here, the flatter simple curve is the 3° curve and $R_1=1{,}910.08$ ft.; the sharper curve is a 5° curve and $R_2=1{,}146.28$ ft.; the given tangent distance is that to the extremity of the sharper curve, and $T_2=404$ ft.; also, $I=35°\ 20'$. Thus, the problem comes under case I.

By formula 1,

$$\begin{aligned}1-\cos I_1&=\frac{T_2\sin I-R_2(1-\cos I)}{R_1-R_2}\\&=\frac{404\sin 35°\ 20'-1{,}146.28\times(1-\cos 35°\ 20')}{1{,}910.08-1{,}146.28}\\&=0.02946\end{aligned}$$

Hence, $\cos I_1=1-0.02946=0.97054$ and $I_1=13°\ 57'$. Ans.

By formula 2,

$$I_2=I-I_1=35°\ 20'-13°\ 57'=21°\ 23'.\ \text{Ans.}$$

(*b*) By formula 3,

$$\begin{aligned}T_1&=(R_1-R_2)\sin I_1-T_2\cos I+R_2\sin I\\&=(1{,}910.08-1{,}146.28)\sin 13°\ 57'-404\cos 35°\ 20'+1{,}146.28\sin 35°\ 20'\\&=517.48\text{ ft. Ans.}\end{aligned}$$

6. Case II.—Given I, R_1, R_2, and T_1; to find I_2, I_1, and T_2. Here, I_2 is found first by a formula derived from formula 5, Art. 3, which is

$$1-\cos I_2=\frac{R_1(1-\cos I)-T_1\sin I}{R_1-R_2} \qquad (1)$$

Then, I_1 is determined by the relation

$$I_1=I-I_2 \qquad (2)$$

and T_2 is calculated by the following formula, which is obtained from formula 6, Art. 3:

$$T_2=R_1\sin I-T_1\cos I-(R_1-R_2)\sin I_2 \qquad (3)$$

7. Case III.—Given I, I_1, R_1, and T_1; to find I_2, R_2, and T_2. For these conditions, the angle I_2 is found by formula 2, Art. 5, and then R_2 is computed from the following formula, which is derived from formula 5, Art. 3:

$$R_1 - R_2 = \frac{R_1(1-\cos I) - T_1 \sin I}{1-\cos I_2}$$

The tangent distance T_2 can be calculated by formula 3 of the preceding article.

EXAMPLE.—Two tangents intersect at an angle of 51° 18′, the station of the point of intersection being 74+65. A compound curve starting at station 68+12 is to be composed of a 4° curve, having a central angle of 30°, and another simple curve. Determine (*a*) the central angle of that simple curve, (*b*) its radius, and (*c*) the length of the tangent to the compound curve at its end.

SOLUTION.—(*a*) The central angle of the second simple curve is

$$I_2 = I - I_1 = 51°18' - 30° = 21°18'. \quad \text{Ans.}$$

(*b*) The tangent distance T_1 is 7,465−6,812=653 ft. and the radius of a 4° curve, or R_1, is 1,432.69 ft. Then,

$$R_1 - R_2 = \frac{R_1(1-\cos I) - T_1 \sin I}{1-\cos I_2}$$

$$= \frac{1{,}432.69 \times (1-\cos 51°\ 18') - 653 \sin 51°\ 18'}{1-\cos 21°\ 18'}$$

$$= 399.56 \text{ ft.}$$

Hence, $R_2 = R_1 - 399.56 = 1{,}432.69 - 399.56 = 1{,}033.13$ ft. Ans.

(*c*) By formula 3, Art. 6,

$$T_2 = R_1 \sin I - T_1 \cos I - (R_1 - R_2) \sin I_2$$

$$= 1{,}432.69 \sin 51°\ 18' - 653 \cos 51°\ 18' - 399.56 \sin 21°\ 18'$$

$$= 564.69 \text{ ft.} \quad \text{Ans.}$$

8. Case IV.—Given I, I_2, R_2, and T_2; to find I_1, R_1, and T_1. Here, the angle I_1 is found by formula 2, Art. 6, and R_1 is calculated by the following formula, derived from formula 2, Art. 3:

$$R_1 - R_2 = \frac{T_2 \sin I - R_2(1-\cos I)}{1-\cos I_1}$$

Then, T_1 is determined by applying formula 3, Art. 5.

In this case, the given radius is that of the sharper curve, whereas in case III, which was explained in the preceding article, the radius of the flatter curve is given. It often happens in practice that the data are for one curve and it is not known whether the other curve is sharper or flatter. In other words, it cannot be told beforehand whether the given radius is R_1 or R_2.

The procedure then is to assume that the given radius is R_1, as in case III, and to apply the formula of Art. 7. If $R_1(1-\cos I) - T_1 \sin I$ comes out positive, the assumption is correct; but, if this quantity is found to have a negative value, the given radius is R_2 and the problem belongs under case IV.

9. Case V. Given I, R_1, T_1, and T_2; to find I_2, I_1, and R_2. In this case, I_2 is determined by formula 7, Art. 3; I_1 by formula 2, Art. 6; and R_2 by the formula in Art. 7.

10. Case VI.—Given I, R_2, T_1, and T_2; to find I_1, I_2, and R_1. For these conditions, I_1 is found by formula 4, Art. 3; I_2 is determined by formula 2, Art. 5; and R_1 is computed by means of the following formula, derived from formula 3, Art. 3:

$$R_1 - R_2 = \frac{T_1 + T_2 \cos I - R_2 \sin I}{\sin I_1}$$

11. Tangent Distances for Compound Curve.—Sometimes the degree and the central angle for each part of a compound curve are known, and it is required to determine the tangent distances. In Fig. 1, AE or EB is the tangent distance for the flatter curve AB, and is equal to $OA \tan \frac{1}{2}I_1$; similarly, BF or FC is the tangent distance for the sharper curve BC, and is equal to $O'B \tan \frac{1}{2}I_2$. Also, $EF = EB + BF$, the angle DEF is equal to AOB, or I_1, and the angle DFE is equal to $BO'C$, or I_2. Hence, in the triangle DEF, the side EF and two angles are known, and the other parts of the triangle can be determined by the principles of trigonometry. Thus, angle $EDF = 180° - DEF - DFE$, and the sides DE and DF are calculated by the relations $DE = \frac{EF \sin DFE}{\sin EDF}$ and $DF = \frac{EF \sin DEF}{\sin EDF}$. Finally, the required tangent distances to the compound curve are

$$AD = AE + DE$$
$$CD = CF + DF$$

Example.—In Fig. 1, AB is a 5° curve whose central angle is 25° and BC is an 8° curve whose central angle is 24°. Calculate the tangent distances AD and CD.

Solution.—The radius of a 5° curve is 1,146.28 ft. and of an 8° curve 716.78 ft. Then,

$$AE = EB = OA \tan \tfrac{1}{2}I_1 = 1{,}146.28 \tan 12^\circ\ 30' = 254.12 \text{ ft.}$$
$$BF = FC = O'B \tan \tfrac{1}{2}I_2 = 716.78 \tan 12^\circ = 152.36 \text{ ft.}$$

Therefore, $EF = EB + BF = 254.12 + 152.36 = 406.48$ ft.

Also, since angle $EDF = 180^\circ - I_1 - I_2 = 180^\circ - 25^\circ - 24^\circ = 131^\circ$,

$$DE = \frac{EF \sin I_2}{\sin EDF} = \frac{406.48 \sin 24^\circ}{\sin 131^\circ} = 219.06 \text{ ft.}$$

and

$$DF = \frac{EF \sin I_1}{\sin EDF} = \frac{406.48 \sin 25^\circ}{\sin 131^\circ} = 227.62 \text{ ft.}$$

The required tangent distances are

$$AD = AE + DE = 254.12 + 219.06 = 473.18 \text{ ft.} \quad \text{Ans.}$$
$$CD = CF + DF = 152.36 + 227.62 = 379.98 \text{ ft.} \quad \text{Ans.}$$

EXAMPLES FOR PRACTICE

1. The curve at the P. C. of a compound curve is a 6° curve with a central angle of 28°, and the curve at the P. T. is a 3° curve with a central angle of 32°. Calculate the length of the tangent: (*a*) through the P. C. and (*b*) through the P. T.

Ans. $\begin{cases} (a)\ 719.11 \text{ ft.} \\ (b)\ 973.77 \text{ ft.} \end{cases}$

2. Two tangents that intersect at an angle of 42° 36′ are to be connected by a compound curve consisting of a 4° curve and a 5° 30′ curve. If the tangent at the beginning of the 4° curve is to be 500 feet long, determine (*a*) the central angle of each curve and (*b*) the length of the tangent at the end of the 5° 30′ curve.

Ans. $\begin{cases} (a)\ I_2 = 26^\circ\ 2',\ I_1 = 16^\circ\ 34' \\ (b)\ 430.30 \text{ ft.} \end{cases}$

3. A compound curve, which consists of an 8° 30′ curve with a central angle of 28° and of another simple curve, connects two tangents that intersect at an angle of 48° 12′. If the intersection of the tangents is at Sta. 39+14 and the 8° 30′ curve begins at Sta. 34+94, what should be (*a*) the radius of the other simple curve and (*b*) the length of the tangent from the P. I. to the P. T. of the compound curve?

Ans. (*a*) 2,107.14 ft.
(*b*) 717.65 ft.

FIELD LAYOUT OF COMPOUND CURVE

12. To lay out a compound curve in the field, it is merely necessary to apply the methods already described for simple curves. The curve *ABC*, Fig. 1, may be located in the field as follows:

First, the point *A* at the beginning of the curve is located either by its station number along the tangent *AD* or by measuring back the proper distance from the point of intersection *D* of the tangents *AD* and *CD*. Then the transit is set up at *A*, the telescope is directed along the tangent *AD* with the vernier reading zero, and the curve *AB* is run in by deflection angles as a simple curve joining the tangents *AE* and *BE*.

The curve *BC* can be located by starting either from *B* or from *C*. If the transit is set up at *B*, a backsight is taken to some point previously established on the curve *AB* with the telescope inverted and the vernier reading the deflection angle from the tangent *BE* for the chord to the point of sight. Care must be taken to turn this deflection angle so that the vernier will read zero when the telescope lies in the tangent *BE*. When the curve *AB*, viewed from *A*, lies to the right of the tangent *AD*, as in Fig. 1, the angle should be turned counter-clockwise from the zero position of the vernier, or so that the zero of the vernier in its final position lies to the right of the zero of the horizontal limb of the transit. If the curve is to the left of the tangent, the angle should be turned clockwise. After the telescope has been directed to the backsight and the lower plate of the transit has been clamped, the telescope is plunged back to normal, the upper clamp is loosened, and the vernier is turned past zero to the deflection angle from the tangent *BF* for the first point on the curve *BC*. Then, that curve is run in by deflection angles from the tangent *BF* just like any other simple curve.

If the curve *BC* is to be run back from *C*, that point is located first by its station number along the tangent *CD* or by its distance from the point *D*. Then, the curve *BC* can be located by deflection angles from the tangent *CD* in the usual manner for a simple curve.

13. When the point *B* is located by turning deflection angles from the tangent *AD* or *CD* and measuring chords of the curve *AB* or *CB*, its position is likely to be inaccurate on account of unpreventable errors in the surveying work. Therefore, the point *B* is often located by its position on the tangent *EF*. The method then consists in measuring from *A* along the tangent *AD* the tangent distance *AE* for the simple curve *AB*; setting up at *E*; turning the angle *DEB*, which is equal to the central angle *AOB*; and laying off the distance *EB* equal to *AE*. The point *B* can also be located by laying off *CF* equal to the tangent distance for the simple curve *BC*, turning the angle *DFB* at *F*, and laying off *FB* equal to *CF*.

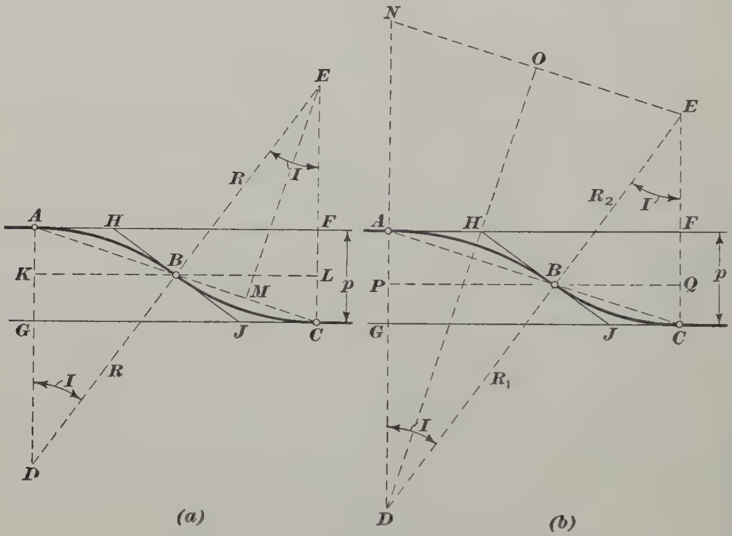
REVERSE CURVES

14. General Remarks.—On a railroad or highway that carries high-speed traffic, reverse curves are dangerous and make riding uncomfortable. Therefore, they should be avoided on such lines, especially where the curvature is sharp. On mountain roads, where two curves that turn in opposite directions must be placed close together, a short tangent should be introduced between the curves. However, on street railways, industrial spur lines, cross-overs, and scenic drives or park roads, reverse curves are sometimes used to advantage.

Reverse curves may be divided into two main classifications: (1) Those in which the tangents at the ends of the reverse curve are parallel; and (2) those in which the end tangents are diverging. In either case, the two simple curves, or branches, of the reverse curve may have radii of the same length or of different lengths.

15. Parallel Tangents Joined by Reverse Curve With Equal Branches.—In Fig. 3, the reverse curve *ABC* consists of the two simple curves *AB* and *BC* that turn in opposite directions and meet at the point *B* where they have a common tangent. This point is called the point of reverse curvature, abbreviated P. R. C. The center of the branch *AB* is at *D*, and the center of the branch *BC* is at *E*. The tangent *AF* through the P. C. and the tangent *CG* through the P. T. are parallel, and the line *HBJ* is tangent to both branches of the curve at the P. R. C. Whether the two branches *AB* and *BC* have equal radii, as in view (*a*), or unequal radii, as in view (*b*), each curve must have the same central angle; in other words, the angles *ADB* and *BEC* must be equal.

For the condition shown in Fig. 3 (*a*) where the two branches of the reverse curve have equal radii, or $DB = BE$, the relation among the radius R, the perpendicular distance p between the tangents through the P. C. and the P. T., and the central angle I of either curve may be derived as follows: Obviously, the P. R. C. at B will be midway between the tangents AF and CG; and, if the line KL is drawn through B and parallel to those tangents, the distances AK and CL will each be equal to $\frac{1}{2}p$. Also, $AK = AD - KD = R - R \cos I = R(1 - \cos I)$. Hence,



(*a*)

(*b*)

FIG 3

$$R(1 - \cos I) = \tfrac{1}{2}p$$

Usually, the distance p and the radius R are known, the degree of curve of each branch of the curve being selected to meet certain specified requirements. Then, the required central angle I for each branch may be found from the relation $1 - \cos I = \frac{\frac{1}{2}p}{R}$, from which either of the following formulas may be obtained:

$$\cos I = \frac{R - \frac{1}{2}p}{R} \qquad (1)$$

$$\text{vers}\, I = \frac{\frac{1}{2}p}{R} \qquad (2)$$

in which I = central angle of each branch of curve;
R = common radius of two branches of curve, in feet;
p = perpendicular distance between tangents through P. C. and P. T., in feet.

16. Sometimes, the data for a reverse curve joining parallel tangents are the perpendicular distance p between the tangents and the length l of the straight line from the P. C. to the P. T. It is then required to determine the radius R and the central angle I for each branch of the curve.

The central angles of the two branches of the curve, as angles ADB and BEC in Fig. 3, are equal. Also, the deflection angle FAB from the tangent AF to the long chord AB and the deflection angle GCB from the tangent CG to the long chord CB are each equal to half of the central angle for the respective branch. Therefore, these two deflection angles are equal to each other, and the chords AB and BC must form a continuous straight line AC from the P. C. to the P. T. If the line EM is drawn perpendicular to AC, it will bisect the chord BC at M. Also, the angles FAC and CEM will be equal, and the right triangles FAC and CEM will be similar. Hence, $EC : CM = AC : CF$, or $R : \frac{1}{4}l = l : p$, and

$$R = \frac{l^2}{4p}$$

in which R = required radius of each branch of reverse curve, in feet;
l = length of straight line from P. C. to P. T., in feet;
p = perpendicular distance between parallel tangents through P. C. and P. T., in feet.

After the radius R has been selected, the central angle I can be found as in the preceding article. Also, the actual length of the straight line from the P. C. to the P. T. is equal to $\sqrt{4pR}$.

EXAMPLE.—Two parallel tangents 20 feet apart are to be joined by a reverse curve with equal branches. If the length of the straight line from the P. C. to the P. T. is to be about 280 feet, what should be (*a*) the degree, based on a 100-foot chord, and (*b*) the central angle of each branch?

Solution.—(*a*) The required radius is approximately

$$R=\frac{l^2}{4p}=\frac{280^2}{4\times 20}=980 \text{ ft.}$$

Probably a 6° curve, whose radius is 955.37 ft., would be selected. Ans.

The actual length of the straight line from the P. C. to the P. T. would then be equal to $\sqrt{4pR}=\sqrt{4\times 20\times 955.37}=276.46$ ft.

(*b*) If a 6° curve is used, $R=955.37$ ft.; and, by formula 1 of the preceding article,

$$\cos I=\frac{R-\frac{1}{2}p}{R}=\frac{955.37-10}{955.37}$$

Hence, $I=8°\ 18'$. Ans.

17. Parallel Tangents Joined by Reverse Curve With Unequal Branches.—In Fig. 3 (*b*), the branches *AB* and *BC* of the reverse curve have unequal radii, and the P. R. C. at *B* is not midway between the parallel tangents *AF* and *CG* through the P. C. and P. T., respectively. However, since the central angles *ADB* and *BEC* of the two branches are equal, the deflection angles *FAB* and *GCB* are also equal and the chords *AB* and *BC* must lie in the same straight line, as for equal branches. Usually, the distance *p* between the tangents is fixed, and the degree of curve of one branch and the length *l* of the straight line from the P. C. to the P. T. are selected to meet the requirements of the particular case. It is then required to determine the radius or degree of the other branch and the central angle for each branch. The formulas to be used in this case may be derived as follows.

If the line *EN* is drawn parallel to the chord *AC*, and the radius *DA* is extended to meet the line *EN* at *N*, then the figure *ACEN* is a parallelogram. Hence, $EN=AC=l$ and $AN=CE$. Also, if the line *DO* is drawn perpendicular to *EN*, it will bisect *EN* at *O*, or $NO=\frac{1}{2}EN=\frac{1}{2}l$. Since the angles *NDO* and *FAC* are equal, the right triangles *NDO* and *FAC* are similar and $DN:NO=AC:CF$. If the radius of the branch *AB* is denoted by R_1 and the radius of the branch *BC* by R_2, $DN=R_1+R_2$ and $(R_1+R_2):\frac{1}{2}l=l:p$. Hence,

$$R_1+R_2=\frac{l^2}{2p} \qquad (1)$$

When the radius of either branch is known, the radius of the other branch can be readily found by applying formula 1.

Also, $\sin FAC=\frac{FC}{AC}$; or, if I represents the central angle of either branch of the curve, as angle NDE, angle FAC is $\frac{1}{2}I$ and

$$\sin \tfrac{1}{2}I=\frac{p}{l} \qquad (2)$$

EXAMPLE.—A reverse curve is made up of two unequal simple curves. The perpendicular distance between the tangents through the P. C. and P. T. is to be 16 feet, the length of the straight line from the P. C. to the P. T. is to be about 320 feet, and one of the branches is to be a 4° curve whose degree is based on a 100-foot chord. Compute (*a*) the required radius of the other branch, and (*b*) the central angle of each branch.

SOLUTION.—(*a*) The radius of the branch whose degree is given will be called R_1 and its value is 1,432.69 ft. Also, in formula 1, $l=320$ ft. and $p=16$ ft. Then,

$$R_1+R_2=\frac{l^2}{2p}=\frac{320^2}{2\times16}=3{,}200 \text{ ft.}$$

and the required radius R_2 of the unknown branch is

$$R_2=3{,}200-1{,}432.69=1{,}767.31 \text{ ft. Ans.}$$

(*b*) By formula 2,

$$\sin \tfrac{1}{2}I=\frac{p}{l}=\frac{16}{320}=0.05 \text{ and } \tfrac{1}{2}I=2° \ 52'$$

Therefore, the required central angle for each branch of the curve is $2\times2°52'=5°44'$. Ans.

18. In case the perpendicular distance p between two parallel tangents and the radii R_1 and R_2 of the two branches of a reverse curve joining the tangents are known, the central angle of either branch may be found in the following manner: In Fig. 3 (*b*), the line PQ is drawn through the P. R. C. at B and parallel to the given tangents AF and CG. Then, if I denotes the central angle for each branch of the reverse curve, and R_1 and R_2 are the radii of the branches AB and BC, respectively,

$AP = R_1\ (1-\cos I)$, $CQ = R_2\ (1-\cos I)$, and $p = AP + CQ = (R_1+R_2)\ (1-\cos I)$. Hence,

$$1-\cos I = \frac{p}{R_1+R_2} \qquad (1)$$

or

$$\text{vers}\, I = \frac{p}{R_1+R_2} \qquad (2)$$

19. Reverse Curve Whose Tangents Are Not Parallel. The various problems that ordinarily arise in determining the required values for reverse curves with non-parallel tangents may be solved by means of three fundamental formulas. In Fig. 4 is shown a reverse curve *ABC* joining the tangents *AV* and *CV*, which are not parallel. In the fundamental formulas, the following notation will be used: the angle *AVC* is represented by I; the radius of the branch *AB*, by R_1; the radius of the branch *BC*, by R_2; the central angle *ADB*, by I_1; the central angle *BEC*, by I_2; the distance *AV*, by T_1; and the distance *CV*, by T_2.

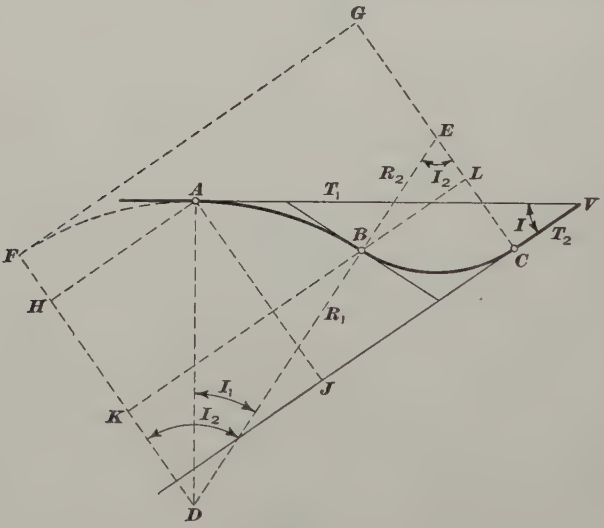


FIG. 4

The first step in the derivation of the formulas is to imagine the branch AB extended backwards beyond the P. C. to a point F at which the tangent FG will be parallel to the tangent CV. Then, FBC is a reverse curve between two parallel tangents, and the central angles FDB and BEC of the two branches are each equal to I_2. Also, as shown in the preceding article, the perpendicular distance between the tangents FG and CV is $CG = (R_1+R_2)\ (1-\cos I_2)$. If AH is drawn perpendicular to the radius DF, then $FH=AD\ (1-\cos ADF)$ or, since $ADF = AVC$, $FH=R_1\ (1-\cos I)$; and, if AJ is drawn perpendicular to the prolongation of tangent VC, then $AJ=AV \sin AVJ = T_1 \sin I$. Since $CG=FH+AJ$,

$$(R_1+R_2)(1-\cos I_2)=R_1(1-\cos I)+T_1 \sin I \qquad (1)$$

Obviously, angle $FDA=FDB-ADB$ or

$$I=I_2-I_1 \qquad (2)$$

If the line KL is drawn through B and parallel to the tangent CV, then $HA+JV=KB+BL+CV$, or

$$R_1 \sin I+T_1 \cos I=R_1 \sin I_2+R_2 \sin I_2+T_2 \qquad (3)$$

20. The general formulas in the preceding article, for reverse curves with non-parallel tangents, are applicable to various problems, but the procedure in solving each type of problem is somewhat different. In the usual problem, the angle between the tangents would be known, but the other data would depend on local conditions. In order to illustrate the use of the fundamental formulas in determining values for a reverse curve, the following typical conditions are assumed: The known values, in addition to the angle I between the tangents, are the radii R_1 and R_2 and the distance T_1 from the P. C. to the vertex; and the values to be determined are the central angles I_1 and I_2 and the distance T_2 from the vertex to the P. T. From formula 1 of the preceding article,

$$1-\cos I_2=\frac{R_1(1-\cos I)+T_1 \sin I}{R_1+R_2} \qquad (1)$$

or

$$\operatorname{vers} I_2=\frac{R_1 \operatorname{vers} I+T_1 \sin I}{R_1+R_2} \qquad (2)$$

Also, from formula 2 of the preceding article,

$$I_1 = I_2 - I \qquad (3)$$

and from formula 3 of the preceding article,

$$T_2 = R_1 \sin I + T_1 \cos I - (R_1 + R_2) \sin I_2 \qquad (4)$$

EXAMPLE.—It is desired to run a reverse curve between two tangents *AV* and *CV*, Fig. 4, which intersect at an angle $AVC = 30°$. The distance from the intersection *V* of these tangents to the P. C. at *A* is to be 1,860 feet, and the radii of the curves *AB* and *BC* are to be 1,909.86 and 1,145.92 feet, respectively. Determine (*a*) the central angles *ADB* and *BEC* of the two branches and (*b*) the distance from the P. T. at *C* to the point *V*.

SOLUTION.—(*a*) By formula 1, in which $R_1 = 1{,}909.86$ ft., $I = 30°$, $T_1 = 1{,}860$ ft., and $R_2 = 1{,}145.92$ ft.,

$$1 - \cos I_2 = \frac{R_1(1 - \cos I) + T_1 \sin I}{R_1 + R_2}$$

$$= \frac{1{,}909.86 \times (1 - \cos 30°) + 1{,}860 \sin 30°}{1{,}909.86 + 1{,}145.92}$$

and $I_2 = 52°\ 16'$. Ans.

Then, by formula 3,

$$I_1 = I_2 - I = 52°\ 16' - 30° = 22°\ 16'. \text{ Ans.}$$

(*b*) By formula 4, the distance *CV* is

$$T_2 = R_1 \sin I + T_1 \cos I - (R_1 + R_2) \sin I_2$$
$$= 1{,}909.86 \sin 30° + 1{,}860 \cos 30° - (1{,}909.86 + 1{,}145.92) \sin 52°\ 16'$$
$$= 149.02 \text{ ft. Ans.}$$

EXAMPLES FOR PRACTICE

1. A reverse curve connects two parallel tangents 20 feet apart. If the total chord distance from the P. C. to the P. T. is to be 380 feet, and the radius of the curve at the P. C. is to be 955.37 feet, what is the required radius of the second curve? Ans. 2,654.63 ft.

2. Two diverging tangents, as *AV* and *CV* in Fig. 4, with an angle of intersection *AVC* of 38°, are to be connected by a reverse curve. The tangent distance *VA* to the P. C. is to be 1,630 feet, and the radii of the branches *AB* and *BC* are to be 818.51 and 954.93 feet, respectively. Determine (*a*) the central angle for the branch *BC*, (*b*) the central angle for the branch *AB*, and (*c*) the tangent distance *CV* to the P. T.

Ans. (*a*) 70° 21′
(*b*) 32° 21′
(*c*) 118.23 ft.

VERTICAL PARABOLIC CURVES

PRELIMINARY EXPLANATIONS

21. **Properties of Parabolic Curves.**—As previously stated, parabolic arcs are in common use for vertical curves on highway and railroad lines. The parabola is a plane curve, which is so formed that every point on it is equally distant from a fixed point called the *focus* and a fixed line called the *directrix*. Thus, in Fig. 5, every point on the parabola ABC is equally distant from the focus F and the directrix DE. If, from any point G on the parabola, a line GF is drawn to the focus F, and a perpendicular GH is dropped to the directrix DE, the distances GF and GH will be equal. Similarly, if IJ is made perpendicular to DE, then $IF = IJ$.

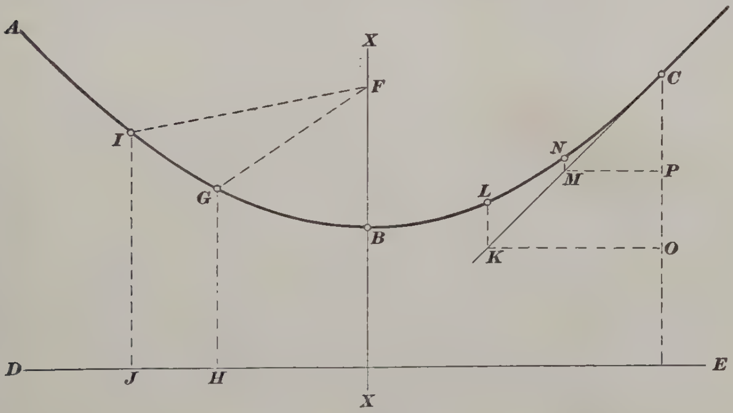


FIG. 5

The line XX, which passes through the focus and is perpendicular to the directrix, is the *axis* of the curve, and the point B, where the curve crosses the axis, is the *vertex*. A parabola may be continued indefinitely, but a parabola for a vertical highway or railroad curve is considered to terminate at the points where the two straight lines connected by the curve are tangent to the curve.

In laying out a parabolic curve, use is made of the fact that the length of a vertical offset from a tangent to a parabola is proportional to the square of the horizontal distance from the point of tangency to that vertical offset. Thus, if the lines KL and MN are vertical offsets from the tangent CK to the curve and the points O and P are in the same vertical line as the point of tangency C, so that KO and MP represent the horizontal distances of the offsets KL and MN from C, then

$$KL : MN = \overline{KO}^2 : \overline{MP}^2$$

22. Conditions at Intersection of Slopes.—The various conditions under which use is made of parabolic vertical curves are illustrated in Fig. 6. In each case the straight slopes or grade lines AV and VB would intersect at V, but a parabolic curve CD is introduced to eliminate the sharp change in slope. For the purposes of determining the elevations of points on vertical curves, the intersections may be divided into two general classes: (1) peaks, or where the curve is below the prolongations of the straight slopes, as in views (a), (b), and (c); and (2) sags, or where the curve is above the prolongations of the straight slopes, as in views (d), (e), and (f).

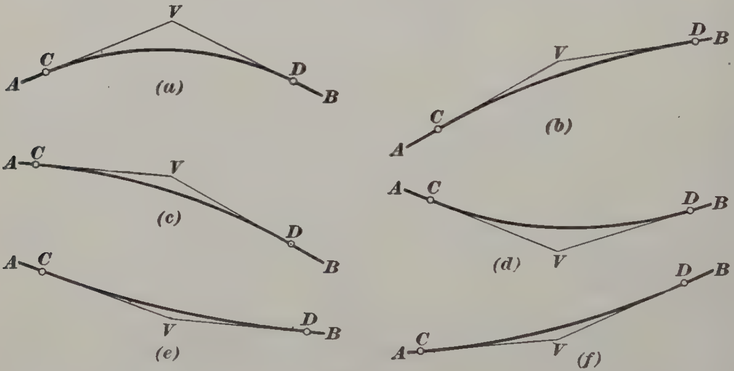


FIG. 6

A vertical curve is generally made to extend an equal horizontal distance on each side of the point of intersection of the slopes. For convenience in performing calculations, this horizontal distance usually is a multiple of 100 feet. However, if certain special requirements are imposed on a vertical curve, it may be necessary to use an unsymmetrical curve with unequal tangents. Thus, when a vertical curve is over a bridge or an overhead crossing and the rates of grade of the approach tangents are already established, an unsymmetrical curve may be required in order to provide the proper clearance for the traffic beneath the bridge.

23. **Rates of Grade.**—The inclination of the grade line of a railroad or highway, or the rate of grade of the road, is usually expressed as a percentage, the value of which is the vertical rise or fall of the line, in feet, in a horizontal distance of 100 feet. If the line slopes uphill in the direction in which the station numbers increase, the rate of grade is said to be plus and the direction of the slope is indicated by the sign +; if, on the other hand, the road slopes downhill in the direction in which the station numbers become greater, the rate of grade is said to be minus and the direction of slope is indicated by the sign −. Thus, if the grade line rises 2 feet in 100 feet, the rate of grade is +2 per cent or, as it is usually written, +2%; if the grade line falls 2 feet in 100, the rate of grade is −2%.

The signs of the rates of grade at an intersection, as well as their amounts, must be considered in computing various values for the vertical curve, and in determining whether the curve will lie above or below the prolongations of the straight slopes. If the algebraic difference between the rate of grade of the left-hand slope and the rate of grade of the right-hand slope is positive, the curve lies below the prolongations of the straight slopes; and if the difference is negative, the curve lies above the prolongations. For example, if the rates of grade of two slopes are +4% and +2%, the algebraic difference is $+4-(+2)=+2\%$, and the curve will be below the prolongations of the slopes, as indicated in Fig. 6 (*b*). Again, if the grades are −3% and −1%, the algebraic difference is $-3-(-1)=-3+1=-2\%$, and the curve will be above the prolongations of the slopes, as in view (*e*).

24. Change in Rate of Grade Along Vertical Curve.—A vertical curve effects a gradual change in the rate of grade, in the same manner as a horizontal curve effects a gradual change in alinement. The grade line of the railroad or highway actually follows the proposed vertical curve, but in computing the locations of the various points on the curve, it is assumed that the grade line is composed of a continuous series of straight lines, or chords, joining successive points on the vertical curve. For instance, in Fig. 7 the slopes *AV* and *VB*, which would actually be joined by a vertical curve extending from *C* to *D*, are assumed to be connected by a series of chords of that curve, such as *CE*, *EF*, and *FG*.

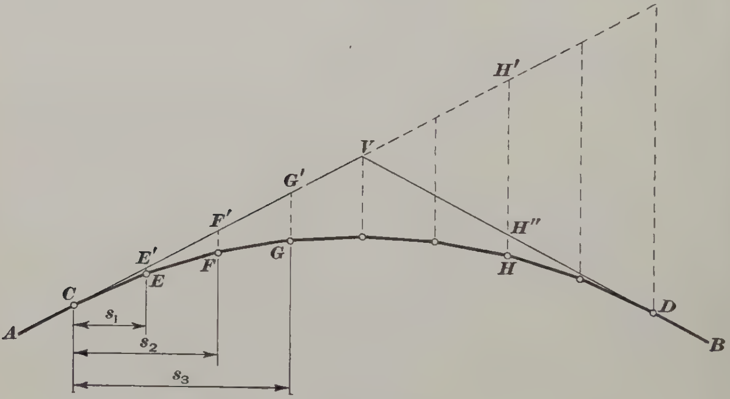


FIG. 7

As previously stated, the lengths of vertical offsets from a straight slope to a vertical parabolic curve are proportional to the squares of the horizontal distances from the point of tangency to the respective offsets. Therefore, the difference between the rate of grade of a slope, as *AV*, and the rate of grade of any chord, as *EF*, of the vertical curve connecting to that slope may be determined as follows: If *a* represents the vertical offset from the slope *AV* at a point that is at a horizontal distance of 100 feet, or 1 station length, from the beginning *C* of the curve, and s_1 is the horizontal distance, in stations of 100 feet, from *C* to any point *E* on the curve, then the vertical offset $E'E$ from the slope to the curve may be found by the relation

$$E'E : a = s_1^2 : 1^2$$

or

$$E'E = s_1^2 a$$

Since s_1 represents the horizontal distance from C to E or E', the difference between the rates of grade of the slope AV and the chord CE is $\frac{E'E}{s_1}$, or

$$\frac{s_1^2 a}{s_1} = a s_1$$

Similarly, if the horizontal distance from C to F is denoted by s_2, the offset from the slope AV at F' is

$$F'F = s_2^2 a$$

Hence, $F'F - E'E = s_2^2 a - s_1^2 a = a(s_2^2 - s_1^2)$. Since the horizontal distance from E to F is equal to $s_2 - s_1$, the difference between the rates of grade of the slope AV and the chord EF is

$$\frac{a(s_2^2 - s_1^2)}{s_2 - s_1} = a(s_2 + s_1)$$

Likewise, if s_3 represents the horizontal distance from C to G, the difference between the rates of grade of the slope AV and the chord FG is equal to

$$a(s_3 + s_2)$$

Therefore, the difference between the rates of grade of the chords EF and FG is

$$a(s_3 + s_2) - a(s_2 + s_1) = a(s_3 - s_1)$$

25. The principles developed by the foregoing explanations may be summarized as follows:

Rule I.—*The difference between the rate of grade of a slope through either end of a vertical curve and the rate of grade of the chord from that end to the first point on the curve is equal to the product of the vertical offset* **a** *(or the offset in feet from the slope to the curve at a horizontal distance of 100 feet from the end of the curve) and the horizontal distance, in stations of 100 feet, between the extremities of the chord.*

Rule II.—*The difference between the rates of grade of any two adjacent chords of a vertical curve is equal to the product of the vertical offset* **a** *and the horizontal distance, in stations of 100 feet, from the beginning of the first chord to the end of the second chord.*

For example, if it is desired to determine the difference between the rates of grade of two adjacent 50-foot chords of a vertical curve for which a is 0.8 foot, the first step is to find the horizontal distance from the beginning of one chord to the end of the other, which is $50+50=100$ feet, or 1 station. Then, by rule II, the required change in the rate of grade is $0.8\times 1 = 0.8\%$.

When two adjacent chords of a vertical curve are each 100 feet long, the difference between their rates of grade is the change in the rate of grade per 100 feet of curve. In this case, the horizontal distance between the opposite ends of the two chords is $100+100=200$ feet, or 2 stations, and therefore for any curve the change in the rate of grade per 100 feet is equal to $2a$.

26. Length of Vertical Curve.—The lengths of vertical curves in highway work are determined by the topography of the ground, the required clear sight distance, and the difference between the rates of grade of the intersecting slopes. At a peak, as in Fig. 6 (a), (b), or (c), the length of curve is generally governed by the sight distance required on the particular road. Much shorter curves are permissible at sags, such as are shown in views (d), (e), and (f). In railroad work, the sight distance is not so important, and the length of any vertical curve is usually determined by the maximum change in rate of grade per 100 feet that is permitted on the particular railroad. Evidently, the total change in rate of grade between the two slopes connected by a vertical curve is equal to the product of the change in rate of grade per 100 feet and the length of the curve, in stations. Conversely, when the rates of grade of the two slopes and the permissible change in rate of grade per 100 feet are known, the approximate length of curve may be found by the formula

$$n=\frac{g-g'}{r}$$

in which n = total length of vertical curve, in stations of 100 feet;

g = rate of grade of first slope;

g' = rate of grade of second slope;

r = permissible change in rate of grade per 100 feet.

The signs of g and g' must be considered in determining the value of $g-g'$, but in finding n the sign of the result is disregarded, and only the numerical value of $g-g'$ is used; in other words, the sign of n is always considered positive.

The length of the curve is generally taken to the next larger even number of stations as 4, 6, 8, etc. Consequently, the actual value of r is never greater than the permissible value and is usually smaller.

EXAMPLE.—Two slopes having grades of −0.4% and +0.18% intersect to form a sag, as in Fig. 6 (*d*). If the rate of change per station must not exceed 0.1%, what should be the length of the vertical curve?

SOLUTION.—Here, $g=-0.4$, $g'=+0.18$, and $r=0.1$. Then, $g-g' = -0.4-(+0.18) = -0.58$ and

$$n=\frac{g-g'}{r}=\frac{0.58}{0.1}=5.8$$

The length of the curve in this case would probably be taken as 6 sta. or 600 ft. Ans.

27. Practical Considerations.—In order to simplify the computations for determining the elevations on a vertical curve, it is desirable to establish the grade lines so that the slopes will intersect at a 100-foot station. Whether or not the intersection is so located, the elevations are generally calculated at the full 100-foot stations; or, if elevations are required at intermediate points, such as the 25-, 50-, or 75-foot stations, those values are also determined. Thus, if the horizontal interval between the points at which elevations are to be determined is 50 feet, and the vertical curve begins at Sta. 18+20 and ends at Sta. 22+20, the elevations would be computed at Stas. 18+20, 18+50, 19+00, 19+50, 20+00, 20+50, 21+00, 21+50, 22+00, and 22+20.

DETERMINATION OF ELEVATIONS ON VERTICAL CURVES WITH EQUAL TANGENTS

METHOD BY OFFSETS FROM SLOPES

28. General Principles of Method.—One method of calculating the elevations of points on a vertical curve is to determine the elevations along the slopes and the lengths of the vertical offsets from the slopes to the curve. Thus, in Fig. 7, the elevation of the point E may be found by taking the difference between the elevation of E' and the length of the vertical offset $E'E$; the elevation of the point F may be obtained from the difference between the elevation of F' and the length of the offset $F'F$; and the elevation of G from the elevation of G' and the length of the offset $G'G$. The elevation of the point H may be found either from the elevation of H', on the prolongation of the slope AV, and the offset $H'H$; or from the elevation of H'', on the slope BV, and the offset $H''H$.

The elevations of points along any slope may be readily computed when the elevation at one point on the slope and the rate of grade of the slope are known. The vertical offsets can be determined in several different ways; the best method depends on the numbers involved in the calculations and the personal preference of the computer.

29. Vertical Offset at Central Point of Curve.—In Fig. 8 are shown typical vertical curves. Whether the curve is at a peak, as in view (a), or at a sag, as in view (b), the following values will usually be known: the rate of grade of each of the slopes AV and VB; the station number and the elevation at the intersection V of the slopes; and the length of the vertical curve CD. Also, unless otherwise specified, it may be assumed that the curve extends equal horizontal distances on each side of the point V.

When it is desired to determine the elevations of points on the left-hand half of the curve by offsets from the slope AV and the elevations of points on the right-hand half by offsets from the slope BV, it is convenient to compute first the offset VE from the intersection of the slopes to the central point on the curve. If the slope AV is prolonged to intersect at D' the vertical line through D, then V is midway between C and D'. Also, if n denotes the horizontal distance, in stations, from C to D, then $VE : D'D = (\frac{1}{2}n)^2 : n^2$, and $VE = \frac{1}{4}D'D$.

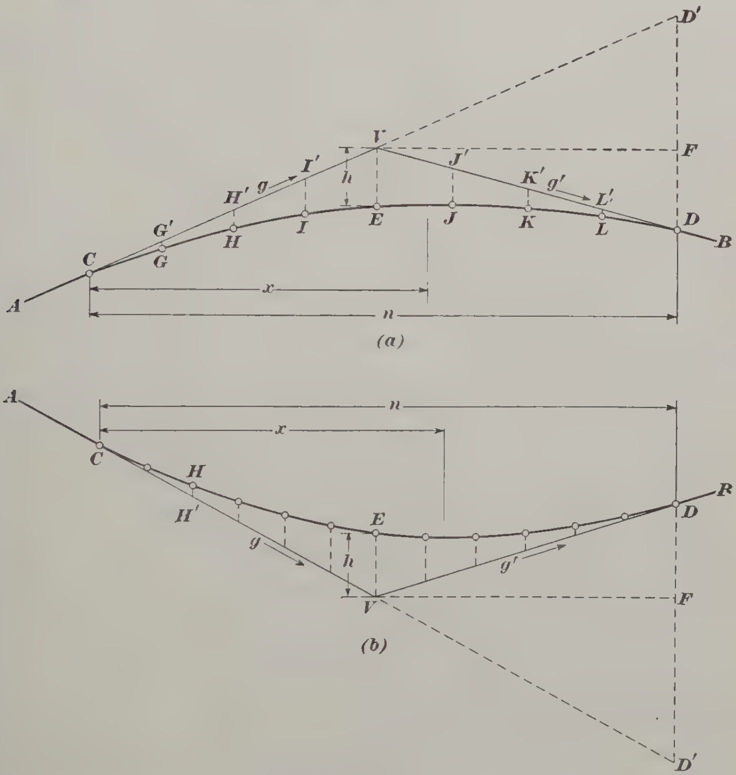


FIG. 8

In order to determine the length of $D'D$, the horizontal line VF is drawn to intersect $D'D$. Then, if g denotes the rate of grade of the slope AV, the distance $D'F$ is equal to $\frac{1}{2}n \times g$; and if g' denotes the rate of grade of the slope BV, the distance FD is equal to $\frac{1}{2}n \times g'$. In Fig. 8 (a) and (b) the rates of grade of the slopes AV and VB have opposite signs, and the distance $D'D$ is the numerical sum of the distances $D'F$ and FD. However, in general, the distance $D'D$ is equal to $\frac{1}{2}n(g-g')$ and the signs of g and g' must be considered in making the calculation. Thus, in Fig. 8 (a) the sign of g is + and the sign of g' is −, and $g-g'$ is a positive number. Since the distance VE is equal to $\frac{1}{4}DD'$, the following general formula is obtained:

$$h=\tfrac{1}{8}n(g-g')$$

in which h = vertical distance, in feet, from point of intersection of slopes to central point on vertical curve;
n = horizontal length of vertical curve, in stations of 100 feet;
g = rate of grade of first slope;
g' = rate of grade of second slope.

Whether the quantity $g-g'$ is positive or negative, the value of h is always considered positive.

30. Vertical Offset at Point 100 Feet From End of Curve. If a represents the vertical offset from the slope AV, Fig 8, to the curve at a horizontal distance of 100 feet from the end C of the curve, and n is the length of the curve, in stations, then $a:D'D=1^2:n^2$, or $a=\frac{D'D}{n^2}$. Since, from the preceding article, $D'D=\frac{1}{2}n(g-g')$, it follows that $a=\frac{\frac{1}{2}n(g-g')}{n^2}$, and the value of a is given by the following formula:

$$a=\frac{g-g'}{2n}$$

in which a = vertical offset, in feet, from slope to curve at horizontal distance of 100 feet from end of curve;
g, g', and n have the same meanings as in the preceding article.

31. Vertical Offset From Slope to Any Point on Curve. When the offset h from the intersection of the slopes to the central point of a vertical curve and the length n of the curve are known, the offset from either slope to any other point on the curve can be readily computed. If the horizontal distance, in stations, from the nearer end of the curve to the point whose elevation is desired is represented by s and the vertical offset, in feet, from the slope to that point by h', then $h' : h = s^2 : (\tfrac{1}{2}n)^2$, and

$$h' = h \times \left(\frac{s}{\tfrac{1}{2}n}\right)^2 \qquad (1)$$

The value of the offset a at a distance of 100 feet from the end of the curve may be determined by substituting 1 for s in this formula; thus, $a = \frac{h}{(\tfrac{1}{2}n)^2}$. Hence, it follows that the value of h' at any point may be found from the value of a by the relation

$$h' = s^2 a \qquad (2)$$

32. Calculation of Elevations on Curve by Offsets From Slopes.—In order to find the elevation of any point on a vertical curve from the elevation of the straight slope at the corresponding station, it is simply necessary to add or subtract the proper vertical offset between the slope and the curve. The offset is subtracted at a peak and added at a sag.

EXAMPLE 1.—Two slopes that intersect at Sta. 61+00 so as to form a peak have grades of +1% and −0.45%. If the elevation of the intersection is 126 feet and the vertical curve is to be 800 feet long, what should be the elevation at each 100-foot station on the curve?

SOLUTION.—The conditions in this case may be represented by Fig. 8 (*a*). Here, the point V is at Sta. 61+00 and, since the curve extends 400 ft. on each side of that point, it starts at Sta. 57+00 and ends at Sta. 65+00. The elevations along the prolongations of the straight slopes from the ends of the curve to the intersection are as follows:

C or Sta. 57, $126 - \frac{400}{100} \times 1 = 122.00$ ft.
G' or Sta. 58, $122 + 1 = 123.00$ ft.
H' or Sta. 59, $123 + 1 = 124.00$ ft.
I' or Sta. 60, $124 + 1 = 125.00$ ft.
V or Sta. 61 $= 126.00$ ft.
J' or Sta. 62, $126 - 0.45 = 125.55$ ft.
K' or Sta. 63, $125.55 - 0.45 = 125.10$ ft.
L' or Sta. 64, $125.10 - 0.45 = 124.65$ ft.
D or Sta. 65, $124.65 - 0.45 = 124.20$ ft.

In this case, $n = 8$ sta., $g = +1\%$, $g' = -0.45\%$, and $g - g' = 1 - (-0.45) = 1 + 0.45 = 1.45$. Hence, by the formula of Art. 29, the value of the offset VE from the intersection of the tangents to the central point of the curve is

$$h=\tfrac{1}{8}n(g-g')=\tfrac{1}{8}\times 8\times 1.45=1.45 \text{ ft.}$$

The difference in elevation between the straight slope and the curve at each station may now be found by either formula of the preceding article. In this example, however, it will be found that the use of formula 2 is not convenient because the value of a would have to be expressed to four decimal places and the computations would be rather laborious. Therefore, the various differences in elevation will be found by formula 1, or

$$h'=h\times\left(\frac{s}{\frac{1}{2}n}\right)^2$$

The horizontal distances s will be assumed to be measured from Sta. 57 for Stas. 58, 59, and 60; and from Sta. 65 for Stas. 62, 63, and 64. The value of $\frac{1}{2}n$ is 4. Obviously, the offset $G'G$ at Sta. 58 will be equal to the offset $L'L$ at Sta. 64; the offset $H'H$ at Sta. 59 will equal the offset $K'K$ at Sta. 63; and the offset $I'I$ at Sta. 60 will equal the offset $J'J$ at Sta. 62. Hence, the values of h' for the various points are:

Stas. 58 and 64, $G'G=L'L=1.45\times(\frac{1}{4})^2=0.09$ ft.
Stas. 59 and 63, $H'H=K'K=1.45\times(\frac{2}{4})^2=0.36$ ft.
Stas. 60 and 62, $I'I=J'J=1.45\times(\frac{3}{4})^2=0.82$ ft.

Since the curve is at a peak, the required elevations are found by subtracting the values of h and h' from the elevations on the straight slope. The computations may be tabulated conveniently as follows:

Station	Elevation on Slope	Correction	Elevation on Curve
57	122.00	0	122.00
58	123.00	0.09	122.91
59	124.00	0.36	123.64
60	125.00	0.82	124.18
61	126.00	1.45	124.55
62	125.55	0.82	124.73
63	125.10	0.36	124.74
64	124.65	0.09	124.56
65	124.20	0	124.20

EXAMPLE 2.—A slope with a grade of −3.2% and one with a grade of +2.6% intersect at Sta. 34+28 at an elevation of 49.20 feet. If a sag is formed by the slopes and they are to be connected by a vertical curve 600 feet long, what should be the elevations of points on the curve at the 50-foot and 100-foot stations?

Solution.—In this case, the conditions are as shown in Fig. 9. The point V is at Sta. 34+28 and its elevation is 49.2 ft. Also, the beginning of the curve at C is 300 ft. from V or at Sta. 31+28, and the other points at which elevations are to be computed are located at Stas. 31+50, 32+00, 32+50, 33+00, 33+50, 34+00, 34+50, 35+00, 35+50, 36+00, 36+50, 37+00, 37+28. Since the slope CV has a rate of grade of -3.2%, the elevation of C is $49.2+\frac{300}{100}\times 3.2=58.8$ ft. The elevation along the slope CV at Sta. 31+50, or 22 ft. from C, is $58.8-\frac{22}{100}\times 3.2=58.096$, or 58.10 ft. The elevation at each succeeding 50-ft. point along that slope may be found by subtracting $\frac{50}{100}\times 3.2=1.6$ ft. from the elevation of the point immediately preceding the one under consideration. Thus, these elevations are: at Sta. 32+00, $58.1-1.6=56.5$ ft.; at Sta. 32+50, $56.5-1.6=54.9$ ft.; at Sta. 33+00, $54.9-1.6=53.3$ ft.; and so on. Similarly, the elevation along the slope VD at Sta. 34+50 is $49.2+\frac{22}{100}\times 2.6=49.77$ ft.; that at Sta. 35+00 is $49.77+\frac{50}{100}\times 2.6=49.77+1.3=51.07$ ft.;

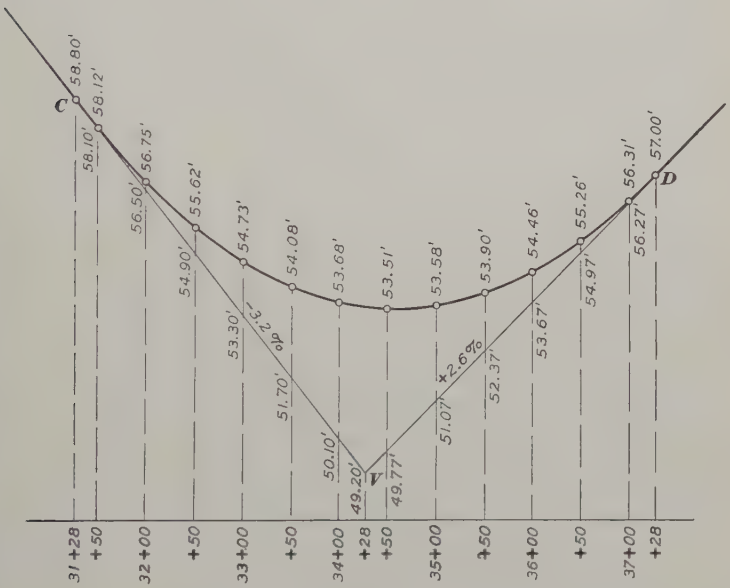


Fig. 9

that at Sta. 35+50 is 51.07+1.3=52.37 ft.; and so on. The elevation at *D*, or Sta. 37+28, is $49.2+\frac{300}{100}\times 2.6=57.0$ ft.

The vertical offset from *V* to the central point on the curve is found by the formula of Art. 29, in which $g-g'=-3.2-(+2.6)=-3.2-2.6=-5.8$. Since the minus sign is neglected in that formula,

$$h=\tfrac{1}{8}n(g-g')=\tfrac{1}{8}\times 6\times 5.8=4.35 \text{ ft.}$$

Then, by formula 1 of the preceding article, in which $\frac{1}{2}n=3$, the differences in elevation between the various points on the slope *CV* and the stations on the curve to the left of the center point are as follows:

$$\text{Sta. } 31+50,\ h'=4.35\times\left(\frac{0.22}{3}\right)^2=0.02 \text{ ft.}$$

$$\text{Sta. } 32+00,\ h'=4.35\times\left(\frac{0.72}{3}\right)^2=0.25 \text{ ft.}$$

$$\text{Sta. } 32+50,\ h'=4.35\times\left(\frac{1.22}{3}\right)^2=0.72 \text{ ft.}$$

$$\text{Sta. } 33+00,\ h'=4.35\times\left(\frac{1.72}{3}\right)^2=1.43 \text{ ft.}$$

$$\text{Sta. } 33+50,\ h'=4.35\times\left(\frac{2.22}{3}\right)^2=2.38 \text{ ft.}$$

$$\text{Sta. } 34+00,\ h'=4.35\times\left(\frac{2.72}{3}\right)^2=3.58 \text{ ft.}$$

Similarly, the differences in elevation between the various points on the slope *VD* and the stations on the curve to the right of the central point are found by taking *s* as the distance from *D*, or Sta. 37+28. As for the points to the left, $h=4.35$ ft. and $\frac{1}{2}n=3$. The values of h' are then as follows:

$$\text{Sta. } 37+00,\ h'=4.35\times\left(\frac{0.28}{3}\right)^2=0.04 \text{ ft.}$$

$$\text{Sta. } 36+50,\ h'=4.35\times\left(\frac{0.78}{3}\right)^2=0.29 \text{ ft.}$$

$$\text{Sta. } 36+00,\ h'=4.35\times\left(\frac{1.28}{3}\right)^2=0.79 \text{ ft.}$$

$$\text{Sta. } 35+50,\ h'=4.35\times\left(\frac{1.78}{3}\right)^2=1.53 \text{ ft.}$$

$$\text{Sta. } 35+00,\ h'=4.35\times\left(\frac{2.28}{3}\right)^2=2.51 \text{ ft.}$$

$$\text{Sta. } 34+50,\ h'=4.35\times\left(\frac{2.78}{3}\right)^2=3.74 \text{ ft.}$$

The elevations of the points on the curve are found by adding the value of h' to the elevations on the slopes CV and VD, as shown in the following tabulation:

Station	Elevation on Slope	Correction	Elevation on Curve
31+28	58.80	0	58.80
31+50	58.10	0.02	58.12
32+00	56.50	0.25	56.75
32+50	54.90	0.72	55.62
33+00	53.30	1.43	54.73
33+50	51.70	2.38	54.08
34+00	50.10	3.58	53.68
34+50	49.77	3.74	53.51
35+00	51.07	2.51	53.58
35+50	52.37	1.53	53.90
36+00	53.67	0.79	54.46
36+50	54.97	0.29	55.26
37+00	56.27	0.04	56.31
37+28	57.00	0	57.00

33. Elevations on Curve by Offsets From Prolongation of One Slope.—When the elevations of all points on a vertical curve are to be computed by offsets from the prolongation of only one slope, the procedure is quite similar to that just described for offsets from two slopes. In this case, however, the distance $D'D$, Fig. 8, is used as a basis for computing all the vertical offsets, and all horizontal distances are taken from the point at which the reference slope is tangent to the vertical curve. As previously shown, the vertical offset, as $D'D$ in Fig. 8, from the prolongation of the reference slope to the other end of the curve may be found by the formula

$$H=\tfrac{1}{2}n(g-g') \qquad (1)$$

in which H = vertical offset, in feet, from prolongation of straight slope through beginning of curve to end of curve on other slope;
n = length of curve, in stations of 100 feet;
g = rate of grade of first slope;
g' = rate of grade of second slope.

Also, the offset from the prolongation of the reference slope to any point on the curve may be found from the relation

$$H' = H \times \left(\frac{S}{n}\right)^2 \qquad (2)$$

in which H' = vertical offset, in feet, from prolongation of reference slope to any point on vertical curve;

S = horizontal distance, in stations of 100 feet, from end of curve on reference slope to point under consideration;

H and n have same meanings as in formula 1.

If a represents the difference in elevation between the prolongation of the slope and the vertical curve at a point 100 feet from the end of the curve on that slope, the value of H' at any other point can be found by the relation

$$H' = S^2 a \qquad (3)$$

The elevations of the points on the curve are found from the elevations on the prolongation of the straight slope by adding or subtracting the proper values of H and H', as described in the preceding article for offsets from two slopes.

EXAMPLE.—Two slopes that intersect at Sta. 28+00 so as to form a peak have grades of +3.5% and +2%. If the elevation of the intersection is 245 feet and the vertical curve is to be 600 feet long, what should be the elevation at each 100-foot station on the curve?

SOLUTION.—The conditions for this case may be represented in Fig. 10. Here, the point V is at Sta. 28+00 and each half of the curve is 300 ft. or 3 sta. long. Hence, the beginning of the curve at C is at Sta. 25+00 and the end at D is at Sta. 31+00. The elevations along the prolongation of the straight slope through Sta. 25+00 are as follows:

Sta. 25, $245 - \frac{300}{100} \times 3.5 = 234.50$ ft.
Sta. 26, $234.5 + 3.5 = 238.00$ ft.
Sta. 27, $238 + 3.5 = 241.50$ ft.
Sta. 28 $= 245.00$ ft.
Sta. 29, $245 + 3.5 = 248.50$ ft.
Sta. 30, $248.5 + 3.5 = 252.00$ ft.
Sta. 31, $252 + 3.5 = 255.50$ ft.

The difference in elevation DE between the prolongation of the slope AV at Sta. 31 and the end D of the curve is found by formula 1. Here, $n = 6$ sta. and $g - g' = 3.5 - (+2) = 1.5$. Hence,

$$H = \tfrac{1}{2} n (g - g') = \tfrac{1}{2} \times 6 \times 1.5 = 4.5 \text{ ft.}$$

In this case, it is convenient to determine the values of H' by means of formula 3 and, therefore, it is necessary first to compute the value of a. By the formula of Art. 30,

$$a=\frac{g-g'}{2n}=\frac{1.5}{2\times 6}=0.125 \text{ ft.}$$

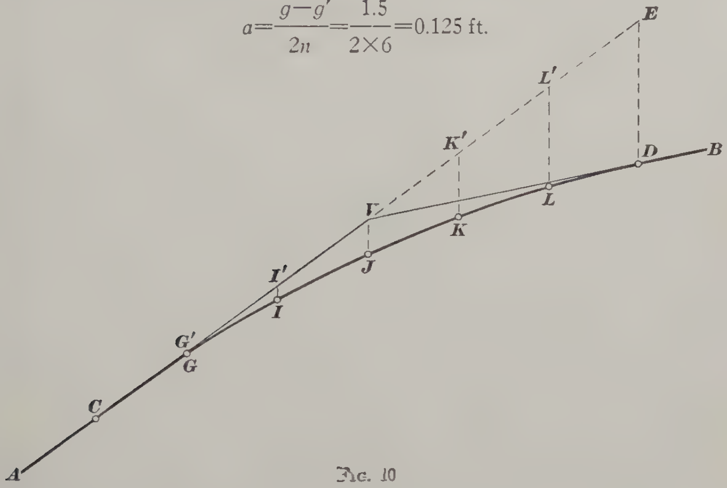


Fig. 10

The value of H' at Sta. 26, or the distance $G'G$, is equal to a, or 0.13 ft., and the other values of H' are as follows:

Sta. 27, $I'I=2^2\times 0.125=0.5$ ft.
Sta. 28, $VJ=3^2\times 0.125=1.13$ ft.
Sta. 29, $K'K=4^2\times 0.125=2.00$ ft.
Sta. 30, $L'L=5^2\times 0.125=3.13$ ft.
Sta. 31, $ED=6^2\times 0.125=4.50$ ft.

Since the curve is at a peak, the required elevations are found by subtracting the values of H' from the elevations on the prolongation of the straight slope. The results of the computations may be tabulated as follows:

Station	Elevation on Slope	Correction	Elevation on Curve
25	234.50	0	234.50
26	238.00	0.13	237.87
27	241.50	0.50	241.00
28	245.00	1.13	243.87
29	248.50	2.00	246.50
30	252.00	3.13	248.87
31	255.50	4.50	251.00

EXAMPLES FOR PRACTICE

1. Two slopes that form a sag have rates of grade of −1.6% and +0.2%, respectively. If they are to be connected by a vertical curve 600 feet long and the elevation of the intersection of the slopes at Sta. 13+00 is 128.66 feet, what should be the elevations at points on the vertical curve that are 100 feet apart?

Ans.

Sta.	Elevation
10	133.46
11	132.01
12	130.86
13	130.01
14	129.46
15	129.21
16	129.26

2. A vertical curve 400 feet long is to connect two slopes forming a peak and having rates of grade of +3.2% and −1.6%, respectively. If the elevation at the intersection of the slopes is 101.4 feet, and the curve begins at Sta. 3+00, determine the elevations on the curve at points 50 feet apart.

Ans.

Sta.	Elevation
3+00	95.00
3+50	96.45
4+00	97.60
4+50	98.45
5+00	99.00
5+50	99.25
6+00	99.20
6+50	98.85
7+00	98.20

3. As indicated in Fig. 10, two intersecting slopes AV and BV, which form a peak, are connected by a 600-foot vertical curve. The grades of AV and BV are +5% and +1.2%, respectively; and the vertex V is at Sta. 43+00 and has an elevation of 137 feet. Determine the elevations of Stas. 41+00 and 44+00 on the curve.

Ans. Sta. 41, 126.68 ft.
Sta. 44, 136.93 ft.

DIFFERENCES IN ELEVATION BETWEEN SUCCESSIVE STATIONS

34. General Features of Method.—Instead of computing the elevations of points on a vertical curve by offsets from the prolongations of the slopes, it is often convenient to determine the elevation of each point on the curve from the elevation of the preceding point. Thus, in Fig. 7, the elevation of E would be found by adding to the elevation of *C* the product of the rate of slope of the chord *CE* and the horizontal distance, in stations, from *C* to *E*. Then, the elevation of *F* would be found by adding to the elevation of *E* the product of the rate of slope of the chord *EF* and the horizontal distance between *E* and *F*; and the elevation of each successive point would be found in a similar manner.

35. Determination of Rate of Slope.—The rates of slope of the various chords may be found by the rules of Art. 25. In the usual case, however, elevations along the curve are determined at regular horizontal intervals. For these conditions, the difference between the rates of grade of any two adjacent chords is the same throughout the curve, and the rate of grade of each chord may be found from that of the preceding chord by subtracting or adding this constant difference.

If the points whose elevations are desired are 100 feet apart, the constant difference is the change in the rate of grade per 100 feet, and its numerical value is equal to $2a$, where a is the vertical offset from the slope to the curve at a horizontal distance of 100 feet from the end of the curve. Also, there is a difference of a between the rates of grade of the straight slope and the 100-foot chord from the end of the curve on that slope to the first station on the curve.

36. Sign of Rate of Slope.—In determining the elevation of a point on a vertical curve from the elevation of the preceding point, it is important to consider the sign of the rate of grade of the chord joining the two points. To obtain the proper sign, it is necessary to consider the sign of the rate of grade of the straight slope through the beginning of the curve. Then, at a peak, the successive differences in rate of grade are subtracted; and at a sag, these differences are added. In every case, the sign of the result is retained.

EXAMPLE 1.—A vertical curve 400 feet long connects two tangents whose rates of grade are, respectively, +0.9% and −0.7%. If the curve starts at Sta. 17+00 and the slopes form a peak, what is the rate of grade of each of the 100-foot chords of the curve?

Solution.—In this case, the points on the vertical curve whose elevations are desired are at horizontal intervals of 100 ft. Also, by the formula of Art. 30,

$$a=\frac{g-g'}{2n}=\frac{0.9-(-0.7)}{2\times4}=0.2 \text{ ft.}$$

Then, the difference between the rates of grade of the slope through Sta. 17 and the chord from Sta. 17 to Sta. 18 is numerically equal to *a* and is 0.2%. Since the curve is at a peak, the rate of grade of that chord is +0.9−0.2=0.7%.

The difference between the rates of slope of two successive 100-ft. chords is numerically equal to $2a$ and is 2×0.2=0.4%. Thus, the rates of grade of the three remaining chords of the curve are as follows:

Sta. 18 to Sta. 19, +0.7−0.4=+0.3%
Sta. 19 to Sta. 20, +0.3−0.4=−0.1%
Sta. 20 to Sta. 21, −0.1−0.4=−0.5%

Example 2.—Two slopes, whose rates of grade are, respectively, −2% and +4%, intersect so as to form a sag. If the slopes are to be connected by a 600-foot vertical curve that starts at Sta. 29+30, and elevations are to be calculated at both the 50-foot and the 100-foot stations, what is the rate of slope of each chord?

Solution.—Here, $g-g'=-2-4=-6\%$ and

$$a=\frac{g-g'}{2n}=\frac{6}{2\times6}=0.5 \text{ ft.}$$

The horizontal distance from Sta. 29+30 to Sta. 29+50 is 20 ft. or 0.2 sta. and by rule I, Art. 25, the difference between the rates of grade of the straight slope through Sta. 29+30 and the chord to Sta. 29+50 is $a\times0.2$ or 0.1%. Since the curve is at a sag, the rate of grade of that chord is −2+0.1=−1.9%.

By rule II, Art. 25, the difference between the rates of grade of the chord from Sta. 29+30 to Sta. 29+50 and the chord from Sta. 29+50 to Sta. 30+00 is equal to the product of *a* and the horizontal distance, in 100-ft. stations, from Sta. 29+30 to Sta. 30+00, or $a\times0.7=0.35\%$. Therefore, the rate of grade of the chord between Stas. 29+50 and 30+00 is −1.9+0.35=−1.55%.

The horizontal distance, in stations, between the opposite ends of two adjacent 50-ft. chords is 0.5+0.5=1, and the difference between the rates of grade of two such chords is equal to $a\times1$ or 0.5%. Thus, the rates of slope of the chords to Sta. 35+00 are as follows:

Stas. 30+00 to 30+50, −1.55+0.5=−1.05%
Stas. 30+50 to 31+00, −1.05+0.5=−0.55%
Stas. 31+00 to 31+50, −0.55+0.5=−0.05%
Stas. 31+50 to 32+00, −0.05+0.5=+0.45%

Stas. 32+00 to 32+50, +0.45+0.5=+0.95%
Stas. 32+50 to 33+00, +0.95+0.5=+1.45%
Stas. 33+00 to 33+50, +1.45+0.5=+1.95%
Stas. 33+50 to 34+00, +1.95+0.5=+2.45%
Stas. 34+00 to 34+50, +2.45+0.5=+2.95%
Stas. 34+50 to 35+00, +2.95+0.5=+3.45%

The difference between the rates of grade of the chords from Sta. 34+50 to Sta. 35+00 and from Sta. 35+00 to Sta. 35+30 is, by rule II, Art. 25, $a \times 0.8 = 0.4\%$. Hence, the rate of grade of the chord between Stas. 35+00 and 35+30 is +3.45+0.4=+3.85%.

37. Elevations on Curve by Differences in Elevation Between Stations.—The elevation of each point on a vertical curve is equal to the elevation of the preceding point plus or minus the difference in elevation between the two points. On any vertical curve, the difference in elevation between two points is equal to the product of the rate of grade of the chord joining the two points and the horizontal distance, in stations, between the two points. If the rate of grade of the chord between the two points is plus, the difference in elevation is added to the elevation of the known point; on the other hand, if the rate of grade of the chord is minus, the difference in elevation is subtracted. When the points whose elevations are required are at 100-foot intervals, the difference in elevation between any two successive points is equal numerically to the rate of grade of the chord joining the two points. In this case, these differences in elevation are computed directly instead of determining first the rates of grade of the chords.

Let g denote the rate of grade of the straight slope through the beginning of a vertical curve; a, the vertical offset from that slope to the curve at a horizontal distance of 100 feet from the point of tangency; and s_1, the horizontal distance, in stations of 100 feet, from the beginning of the curve to the first 100-foot station on the curve. Then, the difference in elevation between the beginning of the curve and a point on the prolongation of the straight slope at the first 100-foot station is $s_1 g$. Also, by formula 2, Art. 31, the vertical offset from the prolongation of the slope to the curve at that station is $s_1^2 a$. Hence, the difference in elevation between the beginning of the curve and the first 100-foot station on the curve is $s_1g-s_1^2a=s_1(g-s_1a)$ at a peak, or $s_1(g+s_1a)$ at a sag. If the curve begins at a 100-foot station, the horizontal distance to the first 100-foot station on the curve is 100 feet, or 1 station, and $s_1=1$. In such a case, the difference in elevation between the beginning of the curve and the first 100-foot station equals $g-a$ at a peak or $g+a$ at a sag.

The difference in elevation between the first and second 100-foot stations at a peak is $s_1\ (g-s_1a)-(1+s_1)\ a$. If the curve starts at a 100-foot station, this expression equals $(g-a)-2a$. At a sag, the minus signs in either expression would be replaced by plus signs. For any curve, the difference in elevation between two consecutive 100-foot stations beyond the second may be obtained from the difference in elevation between the preceding two consecutive stations by subtracting $2a$ at a peak or by adding $2a$ at a sag.

As a check on the results, the elevation of the end of the curve should be determined both by the method given here and by the difference in elevation along the straight slope between the intersection of the slopes and the end of the curve.

EXAMPLE 1.—Determine the elevations of the 100-foot stations on the vertical curve in example 1 of the preceding article, if the elevation of the intersection of the straight slopes at Sta. 19+00 is 186 feet.

SOLUTION.—As stated in the preceding article, the curve is 400 ft. long and starts at Sta. 17+00. Also, the rate of grade of the straight slope through that point is $g=+0.9\%$, and $a=0.2$ ft. The elevation at Sta. 17+00 is $186-\frac{200}{100}\times0.9=184.2$ ft.

Since the curve begins at a 100-ft. station and a peak is formed in this case, the difference in elevation between Stas. 17+00 and 18+00 is $g-a=0.9-0.2=0.7$ ft., and the elevation at Sta. 18+00 is $184.2+0.7=184.9$ ft.

The difference in elevation between Stas. 18+00 and 19+00 on the curve is $0.7-2a=0.7-0.4=0.3$ ft., and the elevation at Sta. 19+00 is $184.9+0.3=185.2$ ft.

The difference between Stas. 19+00 and 20+00 is $0.3-2a=0.3-0.4=-0.1$ ft., and the elevation at Sta. 20+00 is $185.2-0.1=185.1$ ft.

The difference between Stas. 20+00 and 21+00 is $-0.1-0.4=-0.5$ ft., and the elevation at Sta. 21+00, which is the end of the curve, is $185.1-0.5=184.6$ ft.

The elevation at Sta. 21+00, as determined from the elevation of the intersection of the slopes is $186-\frac{200}{100}\times 0.7=184.6$ ft., and therefore all the computed elevations may be assumed to be correct.

The computations for determining the various elevations may be conveniently tabulated as follows:

Station	Difference	Elevation
17+00		184.2
18+00	$g-a=0.9-0.2 = 0.7$	184.9
19+00	$0.7-2a=0.7-0.4 = 0.3$	185.2
20+00	$0.3-0.4 = -0.1$	185.1
21+00	$-0.1-0.4 = -0.5$	184.6

EXAMPLE 2.—If the elevation at Sta. 29+30 in example 2 of the preceding article is 241.75 ft., what are the elevations of the 50- and 100-foot stations on the curve?

SOLUTION.—The rates of grade and the differences in elevation between the successive points whose elevations are desired are as follows:

Stations	Rate of Grade, in Per Cent	Difference in Elevation, in Feet
29+30 to 29+50	−1.9	−0.38
29+50 to 30+00	−1.55	−0.775
30+00 to 30+50	−1.05	−0.525
30+50 to 31+00	−0.55	−0.275
31+00 to 31+50	−0.05	−0.025
31+50 to 32+00	+0.45	+0.225
32+00 to 32+50	+0.95	+0.475
32+50 to 33+00	+1.45	+0.725
33+00 to 33+50	+1.95	+0.975
33+50 to 34+00	+2.45	+1.225
34+00 to 34+50	+2.95	+1.475
34+50 to 35+00	+3.45	+1.725
35+00 to 35+30	+3.85	+1.155

The rates of grade here tabulated were calculated in the preceding article, and each difference in elevation is determined by multiplying the respective rate of grade by the horizontal distance, in stations, between the points in question. Thus, the difference in elevation between Stas. 29+30 and 29+50 is $-1.9\times 0.2=-0.38$ ft., that between Stas.

33+00 and 33+50 is $+1.95\times0.5=+0.975$ ft., and that between Stas. 35+00 and 35+30 is $+3.85\times0.3=1.155$ ft.

The required elevations may be readily determined in the following manner:

Station	Elevation, in Feet
29+30	241.75
29+50	241.75−0.38=241.37
30+00	241.37−0.77=240.60
30+50	240.60−0.53=240.07
31+00	240.07−0.27=239.80
31+50	239.80−0.03=239.77
32+00	239.77+0.23=240.00
32+50	240.00+0.47=240.47
33+00	240.47+0.73=241.20
33+50	241.20+0.97=242.17
34+00	242.17+1.23=243.40
34+50	243.40+1.47=244.87
35+00	244.87+1.73=246.60
35+30	246.60+1.15=247.75

The elevation at Sta. 35+30, as determined from the rates of grade of the slopes, would be $241.75-\frac{300}{100}\times2+\frac{300}{100}\times4=247.75$ ft.

38. Position of High or Low Point.—In the case of a vertical curve joining two tangents that slope in opposite directions, as in Fig. 8, it is sometimes desired to determine the station number and the elevation of the highest point on the curve at a peak or the lowest point at a sag. In either case, the required point is located where the rate of grade becomes zero; that is, where the tangent to the curve is horizontal. Obviously, the horizontal distance x from the beginning of the curve to the point where the rate of grade is zero bears the same ratio to the length of the entire curve as the rate of grade of the straight slope through the beginning of the curve bears to the total difference in the rates of grade of the two straight slopes. Thus, $x:n=g:(g-g')$, or

$$x=\frac{gn}{g-g'}$$

in which x = horizontal distance, in stations of 100 feet, from beginning of vertical curve to high or low point;

g = rate of grade of straight slope through beginning of curve;

n = length of vertical curve, in stations;

g' = rate of grade of straight slope through end of curve.

The elevation of the high or low point can be determined from the elevation of the preceding regular elevation point, the rate of grade of the chord between these two points, and the horizontal distance between the points.

Example 1.—Find (*a*) the station number and (*b*) the elevation of the highest point of the curve in example 1 of Arts. 36 and 37.

Solution.—(*a*) From Art. 36, $g=+0.9$, $g'=-0.7$, and $n=4$. Hence,

$$x=\frac{gn}{g-g'}=\frac{0.9\times4}{0.9-(-0.7)}=2.25 \text{ sta.}$$

The high point is situated 225 ft. from Sta. 17+00, and is at Sta. 19+25. Ans.

(*b*) By rule II, Art. 25, the difference between the rates of grade of the chords from Sta. 18+00 to Sta. 19+00 and from Sta. 19+00 to Sta. 19+25 is $a\times1.25$ or $0.2\times1.25=0.25\%$. From Art. 36, the rate of grade of the former chord is $+0.3\%$; and therefore the rate of grade of the latter chord is $0.3-0.25=+0.05\%$. As found in Art. 37, the elevation of Sta. 19+00 is 185.2 ft. Since the horizontal distance between Stas. 19+00 and 19+25 is 0.25 sta., the elevation at Sta. 19+25 is $185.2+0.05\times0.25=185.21$ ft. Ans.

Example 2.—Find (*a*) the station number and (*b*) the elevation of the lowest point on the curve in example 2 of Arts. 36 and 37.

Solution.—(*a*) In this case, $g=-2$, $g'=+4$, and $n=6$. Therefore,

$$x=\frac{gn}{g-g'}=\frac{-2\times6}{-2-4}=2 \text{ sta.}$$

and, since the beginning of the curve is at Sta. 29+30, the low point is at Sta. 31+30. Ans.

(*b*) The difference between the rates of grade of the chords from Sta. 30+50 to Sta. 31+00 and from Sta. 31+00 to Sta. 31+30 is $a\times0.8$ or $0.5\times0.8=0.4\%$. Since the rate of grade of the former chord is -0.55%, that of the latter chord is $-0.55+0.4=-0.15\%$. Also, from Art. 37, the elevation at Sta. 31+00 is 239.80 ft. Finally, since the horizontal distance between Stas. 31+00 and 31+30 is 0.3 sta., the elevation at Sta. 31+30 is $239.80-0.15\times0.3=239.75$ ft. Ans.

EXAMPLES FOR PRACTICE

1. Solve example 1 on page 36 by the method of differences in elevation between successive stations.

2. Solve example 2 on page 36 by the method of successive differences between stations.

3. Determine (*a*) the station number and (*b*) the elevation of the lowest point of the curve in example 1.

Ans. $\begin{cases} (a)\ 15+33 \\ (b)\ 129.19 \text{ ft.} \end{cases}$

VERTICAL CURVES WITH UNEQUAL TANGENTS

39. Type of Curve.—Where a vertical curve must have a particular elevation at a certain station, and the grades of the tangents have already been established, it is usually necessary to use a curve with unequal tangents. One method of constructing such a curve is given by E. L. Pavlo in *Civil Engineering* for March, 1935. In this method, a compound parabola is used; one parabola extends from the P. C. to a point vertically opposite the point of intersection of the tangents, and a second parabola extends from that point to the P. T. In order that the entire curve between the P. C. and the P. T. will be smooth and continuous, the two parabolas are so constructed that they have a common tangent at the point where they intersect.

40. Vertical Offset at Point of Intersection.—In Fig. 11 is shown a vertical curve with unequal tangents. As in the case of symmetrical vertical curves, the following values will usually be known: the rate of grade of each of the slopes AV and BV, the station number and the elevation at the intersection V of the slopes, and the minimum allowable length of the vertical curve CD. In addition, the minimum allowable elevation at a certain station on the curve is generally specified. For example, if the curve forms an overhead crossing, the required clearance for the underpass will determine the minimum elevation of the curve; the clearance is commonly measured vertically above the street curb nearer the tangent with the steeper slope.

In order to determine the elevations of stations on the compound parabola, it is necessary to compute first the vertical offset to the curve from the point of intersection of the two tangents. On each side of the point of intersection, or for each individual parabola, the length of a vertical offset from the tangent to the curve is proportional to the square of the horizontal distance from the point of tangency to the vertical offset. Thus, in Fig. 11, $EV : FG = \overline{CH}^2 : \overline{CI}^2$; by substituting for EV, FG, CH, and CI the distances h, h_1', n_1, and s_1, respectively, the relation becomes $h : h_1' = n_1^2 : s_1^2$. Then

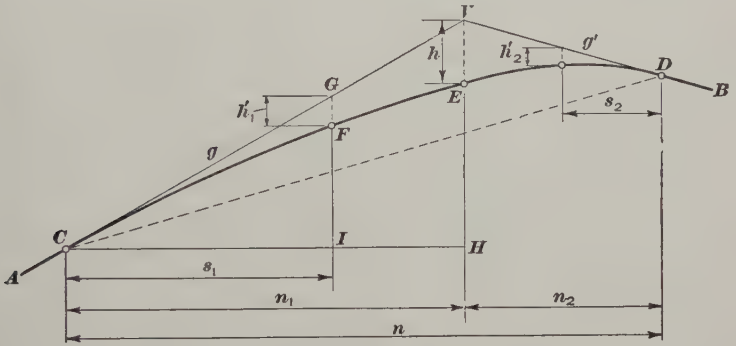


FIG. 11

$$h = h_1'\left(\frac{n_1}{s_1}\right)^2 \qquad (1)$$

in which h = vertical offset, in feet, from point of intersection of tangents to curve;

h_1' = vertical offset, in feet, from tangent to curve at any point between P. C. and point of intersection of tangents;

n_1 = horizontal distance, in 100-foot stations, from P. C. to point of intersection of tangents;

s_1 = horizontal distance, in 100-foot stations, from P. C. to vertical offset h_1'.

Similarly, for the portion of the curve between the point of intersection of the tangents and the P. T.,

$$h = h_2'\left(\frac{n_2}{s_2}\right)^2 \qquad (2)$$

in which h has same meaning as in formula 1;

h_2' = vertical offset, in feet, from tangent to curve at any point between P. T. and point of intersection of tangents;

n_2 = horizontal distance, in 100-foot stations, from P. T. to point of intersection of tangents;

s_2 = horizontal distance, in 100-foot stations, from P. T. to vertical offset h_2'.

Example.—Two slopes that intersect at Sta. 49+00 so as to form a peak have grades of +4% and −2%. The point of intersection has an elevation of 131.0 feet, and the P. C. of a vertical curve connecting the slopes is at Sta. 46+00. If the curb of a road beneath the curve at Sta. 48+80 has an elevation of 112.9 feet, and the desired clearance above the curb is 14 feet, compute the vertical offset from the point of intersection of the tangents to the curve.

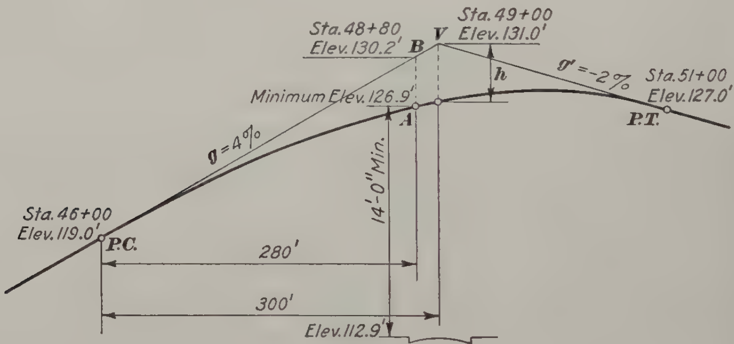


Fig. 12

Solution.—The conditions in this case may be represented by Fig. 12. The point A, which is at Sta. 48+80 on the curve, must have an elevation of at least 112.9+14=126.9 ft. As the rate of grade of the tangent through the P. C. is 4% and Sta. 48+80 is 20 ft. horizontally from the vertex, the elevation of point B on the tangent at Sta. 48+80 is $131.0-4\times 0.2=130.2$ ft. Then, $h_1'=130.2-126.9=3.3$ ft. Also, $n_1=3$, and $s_1=2.8$. From formula 1,

$$h=h_1'\left(\frac{n_1}{s_1}\right)^2=3.3\times\left(\frac{3}{2.8}\right)^2=3.79 \text{ ft. Ans.}$$

41. Length of Compound Parabola.—From the equations of the two parabolas that form a vertical curve with unequal tangents, a formula may be derived for expressing the length of one side of the curve in terms of the length of the opposite side, the vertical offset to the curve from the point of intersection of the tangents, and the rates of grade of the two tangents. Thus, for the values indicated in Fig. 11,

$$n_1=\frac{2hn_2}{n_2(g-g')-2h} \qquad (1)$$

and

$$n_2=\frac{2hn_1}{n_1(g-g')-2h} \qquad (2)$$

in which n_1, n_2, and h have the same meanings as in the preceding article;

g = rate of grade of tangent through P. C.;

g' = rate of grade of tangent through P.T.

The total length n of the curve is the sum of the lengths of the two parts, or

$$n=n_1+n_2 \qquad (3)$$

The computed length of the curve may have to be altered in order to obtain suitable tangent lengths. If the length on either side of the intersection of the tangents were increased, a flatter curve would have to be used and the elevations at all stations on the curve would be decreased. Since such a change would not permit the provision of the required clearance at a particular station, any adjustment in the length of the curve must be a shortening. The corrected value of the vertical offset h at the point of intersection will then be less than the value previously calculated. This new value of h may be found by the formula

$$h=\frac{n_1n_2}{2(n_1+n_2)}(g-g') \qquad (4)$$

EXAMPLE.—Determine (*a*) the length of the curve in the example of the preceding article, and (*b*) the new length of the vertical offset at the point of intersection.

SOLUTION.—(*a*) In this case, $n_1=3$ sta., $h=3.79$ ft., $g=+4\%$, and $g'=-2\%$. Then, from formula 2,

$$n_2=\frac{2hn_1}{n_1(g-g')-2h}=\frac{2\times3.79\times3}{3[4-(-2)]-2\times3.79}=2.18 \text{ sta.}$$

If it is desired to use a curve with a length in full 100-foot stations, and yet maintain at least a 14-ft. clearance at Sta. 48+80, the length n_2 must be changed to 2 sta., and the total length of curve is

$$n = n_1 + n_2 = 3 + 2 = 5 \text{ sta. or } 500 \text{ ft.} \quad \text{Ans.}$$

(*b*) From formula 4, the new value of the vertical offset h at the point of intersection will be

$$h = \frac{n_1 n_2}{2(n_1 + n_2)}(g - g') = \frac{3 \times 2}{2 \times (3+2)} \times (4+2) = 3.6 \text{ ft.} \quad \text{Ans.}$$

42. Calculation of Elevations on Compound Parabola.

When the vertical offset h from the point of intersection of the tangents to the curve and the lengths n_1 and n_2 of the two parts of the curve are known, the vertical offset from either slope to any other point on the curve can be readily computed by the method explained in Art. 31. However, for a curve with unequal tangents, the offsets for the portions of the curve on either side of the point of intersection must be computed separately. Thus, for the part of the curve between the P. C. and the vertex, the vertical offset at any point may be obtained from the formula

$$h_1' = h\left(\frac{s_1}{n_1}\right)^2 \qquad (1)$$

and for the part of the curve between the P. T. and the vertex, the vertical offset at any point may be obtained from the formula

$$h_2' = h\left(\frac{s_2}{n_2}\right)^2 \qquad (2)$$

in which the letters have the same meanings as in Art. 40.

In order to find the elevation of any point on the curve, it is simply necessary to compute the elevation of the slope at the corresponding station and to subtract the proper vertical offset between the slope and the curve.

Example.—Determine the elevation at each 100-foot station on the curve in the example of Arts. 40 and 41.

Solution.—Here, the vertex V, Fig. 12, is at Sta. 49+00, and has an elevation of 131.0 ft. Since the curve extends 300 ft. to the left of V and 200 ft. to the right, the elevations for the 100-ft. stations along the tangents are as follows:

Sta. 46, $131.0-\frac{300}{100}\times 4=119.0$ ft.
Sta. 47, $119+4 \quad =123.0$ ft.
Sta. 48, $123+4 \quad =127.0$ ft
Sta. 49, $127+4 \quad =131.0$ ft.
Sta. 50, $131-2 \quad =129.0$ ft.
Sta. 51, $129-2 \quad =127.0$ ft.

In this case, the value of the offset h at the point of intersection is 3.6 ft. and, from formula 1, the differences in elevation h_1' between the straight slope and the curve for the stations between the P. C. and the vertex are:

Sta. 47, $3.6\times(\frac{1}{3})^2=0.4$ ft.
Sta. 48, $3.6\times(\frac{2}{3})^2=1.6$ ft.
Sta. 49 $\quad =3.6$ ft.

From formula 2, the difference in elevation between the straight slope and the curve at Sta. 50 is

$$3.6\times(\tfrac{1}{2})^2=0.9 \text{ ft.}$$

Then the required elevations on the curve are as follows:

Station	Elevation on Slope	Correction	Elevation on Curve
46	119.0	0.0	119.0
47	123.0	0.4	122.6
48	127.0	1.6	125.4
49	131.0	3.6	127.4
50	129.0	0.9	128.1
51	127.0	0.0	127.0

43. Position of High Point.—Where a vertical curve with unequal tangents is required, there will usually be a high point on the curve. This point may lie either to the left or to the right of the intersection point of the tangents. If $\frac{2h}{g}$, in Fig. 11, is greater than n_1, or if $\frac{-2h}{g'}$ is less than n_2, the high point occurs to the left of the vertex V.

For a curve whose high point is to the left of the vertex, the distance in 100-foot stations from the beginning of the curve to the high point may be found by the formula

$$x_1 = \frac{gn_1^2}{2h} \qquad (1)$$

in which x_1 = horizontal distance, in 100-foot stations, from left end of curve to high point of curve;

g = rate of grade of slope through left end of curve;

n_1 = horizontal distance, in 100-foot stations, from left end of curve to point of intersection of slopes;

h = vertical offset, in feet, to curve at point of intersection.

Where the high point is to the right of the vertex,

$$x_2 = -\frac{g'n_2^2}{2h} \qquad (2)$$

in which x_2 = horizontal distance, in 100-foot stations, from right end of curve to high point of curve;

g' = rate of grade of slope through right end of curve;

n_2 = horizontal distance, in 100-foot stations, from right end of curve to point of intersection of slopes.

Example.—Determine the station of the high point on the curve in the example of Arts. 40, 41, and 42.

Solution.—Here, $h=3.6$ ft., $g=+4\%$, $g'=-2\%$, $n_1=3$, and $n_2=2$. Then, $\frac{2h}{g}=1.8$, which is less than 3; or $\frac{-2h}{g'}=3.6$, which is greater than 2. Consequently, the high point must be to the right of the point of intersection of the slopes.

From formula 2,

$$x_2 = -\frac{g'n_2^2}{2h} = -\frac{-2\times 2^2}{2\times 3.6} = 1.11 \text{ sta.}$$

Since the P. T. is at Sta. 51+00, and 5,100−111=4,989 ft., the high point is at Sta. 49+89. Ans.

EXAMPLES FOR PRACTICE

1. A slope with a grade of +1.5% intersects a slope with a grade of −3% at Sta. 18+00. The elevation of the point of intersection is 473.0 feet and the P. C. of a vertical curve connecting the two slopes is at Sta. 16+00. If the elevation of the curve at Sta. 17+16 must be at least 470.64 feet to allow proper clearance for a road below, what will be

(*a*) the maximum allowable vertical offset from the point of intersection of the tangents to the curve, and (*b*) the theoretical length, in stations, of the right-hand side of the curve?

Ans. (*a*) 3.27 ft.
(*b*) 5.32 sta.

2. If the total length of the curve in the preceding example is made 700 feet, what will be (*a*) the corrected vertical offset from the intersection of the tangents to the curve and (*b*) the station, to the nearest foot, of the high point on the curve?

Ans. (*a*) 3.21 ft.
(*b*) Sta. 16+93